NUMERICAL METHODS
OF ANALYSIS
IN ENGINEERING

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NUMERICAL METHODS OF ANALYSIS IN ENGINEERING

(Successive Corrections)

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Dedicated to Hardy Cross whose simple demonstration of the power of numerical analysis brought these methods within the horizon of practicing engineers.

PREFACE

When a solution is impracticable, an engineer must be prepared to devise a satisfactory approximation. This approach, so inherent to the practice of engineering, was formalized in the Hardy Cross procedure of moment distribution. Engineers sensed that Cross was only organizing in a formal way the procedures of approximation that they had intuitively been using when rigorous mathematical analysis proved impractical. Hence, although methods of successive corrections had been used in mathematical analysis long before Hardy Cross — at least one method was used by Newton — it is clearly true that Cross provided the stimulus that gave rise to a tidal wave of interest in numerical procedures during the past twenty years.

The development that followed the announcement of moment distribution has taken a forked path. Those interested in mathematical applications have shown that the Cross procedure can be set up as the solution by iteration of a set of simultaneous equations and proved mathematically to converge. But Cross did not develop the method in a mathematical way, did not present it as such, and practicing engineers have found its physical emphasis to be one of its most significant contributions to design. Through twenty years of development of the tools of numerical analysis this fork in the path has failed to converge, but instead, mathematical and physical lines of progression have continued almost independently.

In its early development numerical analysis was applied mainly to problems where the geometry consisted merely of intersecting lines. Then Southwell in England and others in America became interested in numerical solutions for two-dimensional plate problems. These problems include stresses in walls and plates, heat flow or electrical flow in plates, and heat flow, fluid flow, or electrical discharge through a gas where the medium is confined between parallel walls so that the phenomena is largely two-dimensional. Whether all of these problems have been solved by numerical methods as yet, I do not know, but they bear a similarity indicating that they could be solved by related numerical procedures.

To solve the continuous two-dimensional problem required an analogy. The two-dimensional space had to be represented by a lattice with intersecting lines forming nodes. Then the problem returned to the lineal geometry that had already succumbed to numerical procedures. The primary difficulty encountered was the tedious repetition of relaxing joints or permitting heat flow past nodes or otherwise successively approaching a state of equilibrium. Since ten subdivisions in each direction of a plane area result in one hundred joints or nodes, the accounting problem can clearly become serious. Hence, throughout the literature one finds attempts to reduce the labor of the analysis. Several of the methods presented in the following chapters are further contributions toward simplification. And in these chapters as in the literature there will be found a division of emphasis between the objective of obtaining an approximation of the function that will solve a differential equation, which is the mathematical approach, or the objective of obtaining at once through physical convergence the value of the stress, deflection, temperature, or potential desired.

Numerical procedures have a theoretical application to three-dimensional problems as well. But since the lineal division of one dimension into ten parts results in one thousand internal nodes in a cube that has ten subdivisions on each side, the practicality of numerical solutions of space problems seems relegated to the future. Such a solution would seem to depend upon joining the simple approach of relaxation and distribution with the mathematical machine operations that have recently become available. It is conceivable that electronic machines may be devised that will perform thousands of distributions in a few minutes and thus adequately solve three-dimensional problems of elasticity and flow.

It is fortunate indeed that it is possible to honor Hardy Cross for his unique contribution to the development of numerical methods of analysis without in any way failing to recognize the great contributions of those who preceded and of those who have followed Cross. Most of the large numbers of references included in this book are a process of extending credit to the many contributors to this field. Cross' contribution when first made, being essentially nonmathematical and emphasizing tangible or physical qualities such as one finds in the term 'slope' as contrasted to the nonphysical or intangible feeling associated with a partial derivative of a higher order, was of a unique kind. It therefore seems appropriate at this time to acknowledge the

twenty-fifth anniversary of the birth of the idea of moment distribution.

The chapters of this book were not prepared by their authors simply to present the results of their researches but to teach methods and effective techniques of numerical analysis. The chapter order, from the study of specific methods to general comparisons of interrelated methods, was chosen for the same reason. The methods and procedures discussed are applicable in many fields of engineering and science. One who attacks the problem seriously will find it possible to use the methods discussed here in fields where numerical analysis has received less attention than in Mechanics.

It is a pleasure to acknowledge the aid of my associates who constituted an informal committee for the symposium: W. A. Casler, Lloyd Donnell, E. I. Fiesenheiser, Max Frocht, J. E. Goldberg, Le Van Griffis, H. V. Hawkins, H. T. Heald, T. J. Higgins, R. L. Janes, Max Jakob, E. F. Masur, S. F. Musselman, Eli Sternberg, R. L. Stevens.

L. E. Grinter

Chicago, Illinois
January, 1949

CONTENTS

THE CHAPTERS AND THEIR AUTHORS

Chapter 1. Hardy Cross published his article entitled "Analysis of Continuous Frames by Distributing Fixed End Moments" in Transactions, ASCE, 1932. It is reprinted here because it is clear that the influence of this short publication produced the impetus that thrust numerical procedures into the consciousness of engineers and led to their widespread development and to their use in engineering practice. In Transactions ASCE, Vol. 96, 1932, it will be found that this paper of 10 pages is followed by 146 pages of discussion by 38 authors. Its effect upon engineering thought in structural mechanics was instantaneous. When he prepared this paper Hardy Cross was Professor of Structural Engineering at the University of Illinois. He is now Chairman of the Department of Civil Engineering at Yale University.

Chapter 2. Linton E. Grinter is Research Professor of Civil Engineering and Mechanics at Illinois Institute of Technology. Having worked with Professor Cross on the early applications of moment distribution, Dr. Grinter since the early 1930's has published books and articles emphasizing the physical characteristics, as contrasted to mathematical iteration, of his and Cross' procedures of relaxation and distribution. During this period work has also gone forward on the application of the author's concepts to problems in elasticity, although this is a first publication of his method for the analysis of the plane state of stress by grid analogy. In general, this chapter advances the author's belief that the best tool of successive corrections is the one that departs least from a study of the physical deformations of the body acting under the loads. As an appendix a problem on the numerical analysis of a notched beam, solved by Andrew J. Pyka, is included.

Chapter 3. F. S. Shaw is Senior Research Officer, Council for Scientific and Industrial Research, Division of Aeronautics, Melbourne, Australia. After a period of association with R. V. Southwell in England, he came to the Graduate Division of Applied Mathematics of Brown University. In this paper the author deals with problems that may be expressed by two-dimensional linear partial differential equations of the boundary-value type. He explains how these equa-

tions may be replaced, in effect, by a large number of algebraic simultaneous equations which are then solved numerically by the Southwell procedure of relaxation. Such devices as block operators and curved boundary operators are treated. It is shown that the methods discussed are of general application.

Chapter 4. R. V. Southwell is Rector of the Imperial College, London, England. Dr. Southwell's research for more than a decade has been directed to the development of his well-known procedures of relaxation. This work has resulted in two books presenting applications of relaxation procedures to problems in engineering and in theoretical physics. This chapter investigates possible approaches to higher accuracy in numerical analyses. The author contrasts his preferred approach through the use of a finer net with the mathematician's preference for an improved approximation in the expression of the finite differences without reduction of the spacing between lines of the net. This results in an interesting set of comparisons.

Chapter 5. M. M. Frocht is Research Professor of Mechanics at Illinois Institute of Technology. His research work in photoelasticity and his extensive publications in that field have made him widely known. Since 1940 he has interested himself in the method described in this chapter which deals with a useful simplification in the numerical solution of Laplace's equation by iteration. Specifically, a simple procedure, called the Linear Rosette method, is proposed for rapid determination of initial values. The rosette may be viewed as an intersecting set of taut strings whose slopes are found from known boundary values of the harmonic function. The effect is to reduce markedly the number of iterations necessary to obtain approximate convergence.

Chapter 6. L. M. K. Boelter is Dean of the College of Engineering of the University of California at Los Angeles. He has been active for many years in advancing research in heat transfer both by experiment and by analysis. The co-author, Myron Tribus, is Instructor in Engineering in the same institution. He began his association with heat transfer research through the Wright Field studies of the Air Corps. This chapter presents a numerical counterpart of the Schmidt-Binder graphical method of finite differences. If it is desired to know the temperatures in a solid as a function of the initial and surface temperatures, the proposed short cut will be useful. However, it will be of greatest utility when it is desired to ascertain the effect of differing boundary and initial temperatures upon the temperature at a particular interior position.

Chapter 7. G. M. Dusinberre is Professor in the Department of Mechanical Engineering of Pennsylvania State College. His interest in numerical methods extends into several fields. His writings have done much to clarify the use of numerical methods in heat transfer. This chapter treats one transient and one steady state problem in the heating of buildings. These problems, solved elsewhere in the literature, are used to demonstrate practical numerical methods in this field and to present comparisons as to the facility and accuracy with which calculations can be made. In an appendix, the method discussed is applied by J. D. Bottorf and Y. S. Touloukian to a problem on temperature distribution through the ground.

Chapter 8. Frank Baron is Professor of Civil Engineering at Northwestern University. He has previously published a paper on the numerical solution of the torsion problem of multiconnected thin-walled cylinders. This chapter compares the approach to numerical solutions in such related fields as structures, vibrations, and elasticity. Analogies which result from requirements similar to those of statics and geometry in structural mechanics are discussed. Various numerical procedures are classified according to the order observed in meeting the controlling requirements. Such procedures are also classified as to the sequence followed in the numerical steps of analysis. Advantages and disadvantages of these different approaches are compared.

Chapter 9. N. M. Newmark is Research Professor of Civil Engineering at the University of Illinois. His research publications have covered both experimental investigations and theoretical analyses of such complex structures as reinforced concrete bridge floors. In this chapter the author surveys many approaches to numerical methods of analyzing stresses, strains, and displacements in structures under static and dynamic loading. He compares iteration with step-by-step calculations, relaxation with continuity restoration, collocation with least squares, etc. Plane stress problems with their lattice analogies, computation of influence coefficients, finer nets, extrapolation, and the use of modified beam loads all receive attention. A balance is maintained between physical and mathematical viewpoints.

Chapter 10. Thomas J. Higgins is Professor of Electrical Engineering at the University of Wisconsin. His published research has been divided between the fields of mechanics and electrical engineering. He has published a number of bibliographies and has an extremely wide acquaintance with the literature of mechanics. This chapter concludes with a comprehensive bibliography of 140 references. The

author compares the variational methods of Galerkin, Kantorovic, Rayleigh-Ritz, Trefftz, and others with the finite difference methods of Collatz, Cross, Liebmann, and Southwell. The basic differences in these procedures are illustrated by their applications to the classical problem of the torsion of a prism. The emphasis here is usually upon approximating the mathematical function that will solve the variational problem or satisfy the partial differential equation of the boundary-value problem.

CHAPTER 1

ANALYSIS OF CONTINUOUS FRAMES BY DISTRIBUTING FIXED-END MOMENTS *

HARDY CROSS **

Synopsis. The purpose of this paper is to explain briefly a method which has been found useful in analyzing frames which are statically indeterminate. The essential idea which the writer wishes to present involves no mathematical relations except the simplest arithmetic. It is true that in order to apply the method it is necessary to determine certain constants mathematically, but the means to be used in determining these constants are not discussed in the paper, nor are they a part of the method. These constants have been derived by so many writers and in so many slightly different ways that there is little occasion to repreat here the whole procedure.

The reactions in beams, bents, and arches which are immovably fixed at their ends have been extensively discussed. They can be found comparatively readily by methods which are more or less standard. The method of analysis herein presented enables one to derive from these the moments, shears, and thrusts required in the design of complicated continuous frames.

Definitions. For convenience of reference, definitions of three terms will be introduced at once. These terms are "fixed-end moment," "stiffness," and "carry-over factor."

By "fixed-end moment" in a member is meant the moment which would exist at the ends of the member if it ends were fixed against rotation.

"Stiffness," as herein used, is the moment at one end of a member (which is on unyielding supports at both ends) necessary to produce unit rotation of that end when the other end is fixed.

If one end of a member which is on unyielding supports at both ends is rotated while the other end is held fixed, the ratio of the moment at

* Published in May, 1930, *Proceedings* and Vol. 96, 1932, *Transactions*, ASCE.

** Then Professor of Structural Engineering, University of Illinois; now Chairman of Civil Engineering, Yale University.

the fixed end to the moment producing rotation at the rotating end is herein called the "carry-over factor."

Effect of Joint Rotation. Imagine any joint in a structure, the members of which are being deformed by loads, or in some other way, to be first held against rotation and then released. Call the algebraic sum of the fixed-end moments at the joint the "unbalanced fixed-end moment." Before the joint is released this unbalanced fixed-end moment will not usually be zero; after the joint is released, the sum of the end moments at the joint must be zero. The total change in end moments, then, must equal the unbalanced fixed-end moment. This may be stated in another way by saying that the unbalanced fixed-end moment has been "distributed to" the connecting members in some ratio.

When the joint is released all connecting members rotate through the same angle, and this rotation at the end is accompanied by a change in end moment. The change in end moments is proportional to the "stiffness" of the members.

Hence, it may be said that when the joint is released the unbalanced fixed-end moment is distributed among the connecting members in proportion to their stiffness.

The rotation of the joint to produce equilibrium induces moments at the other ends of the connecting members. These are equal in each member to the moments distributed at the rotating joint multiplied by the carry-over factor at the rotating end of the member. This follows from the definition of "carry-over factor."

Moment Distribution. The method of moment distribution is this: (*a*) Imagine all joints in the structure held so that they cannot rotate and compute the moments at the ends of the members for this condition; (*b*) at each joint distribute the unbalanced fixed-end moment among the connecting members in proportion to the constant for each member defined as "stiffness"; (*c*) multiply the moment distributed to each member at a joint by the carry-over factor at that end of the member and set this product at the other end of the member; (*d*) distribute these moments just "carried over"; (*e*) repeat the process until the moments to be carried over are small enough to be neglected; and (*f*) add all moments — fixed-end moments, distributed moments, moments carried over — at each end of each member to obtain the true moment at the end.

To the mathematically inclined the method will appear as one of solving a series of normal simultaneous equations by successive approx-

imation. From an engineering viewpoint it seems simpler and more useful to think of the solution as if it were a physical occurrence. The beams are loaded or otherwise distorted while the joints are held against rotation; one joint is then allowed to rotate with accompanying distribution of the unbalanced moment at that joint and the resulting moments are carried over to the adjacent joints; then another joint is allowed to rotate while the others are held against rotation; and the process is repeated until all the joints are "eased down" into equilibrium.

Beam Constants. This method of analysis is dependent on the solution of three problems in the mechanics of materials. These are the determination of the fixed-end moments, of the stiffness at each end, and of the carry-over factor at each end for each member of the frame under consideration. The determination of these values is not a part of the method of moment distribution and is not discussed in this paper.

The stiffness of a beam of constant section is proportional to the moment of inertia divided by the span length, and the carry-over factor is $-\frac{1}{2}$.*

The proof or derivation of these two statements and the derivation of formulas for fixed-end moments is left to the reader. They can be deduced by the use of the calculus; by the theorems of area-moments; from relations stated in *Bulletin 108* of the Engineering Experiment Station of the University of Illinois (the Slope-Deflection *Bulletin*); from the theorem of three moments; by what is known to some as the column analogy method; ** or by any of the other corollaries of geometry as applied to a bent member. Formulas for fixed-end moments in beams of uniform section may be found in any structural handbook.

Signs of the Bending Moments. It has seemed to the writer very important to maintain the usual and familiar conventions for signs of bending moments, since these are the conventions used in design.

For girders the usual convention is used, positive moment being such as sags the beam. For vertical members the same convention is applicable as for girders if the sheet is turned to read from the right as vertical members on a drawing are usually read. The usual conventions for bending moments are, then, applicable to both girders

* The negative carry-over factor applies with a fiber-stress sign convention for bending moment. For a convention of signs dependent upon direction of the curved arrow representing an end moment, the carry-over factor is positive. Distribution is often performed with the positive sign indicating clockwise action upon the adjacent joint. *Ed.*

** Hardy Cross, *The Column Analogy*, University of Illinois Eng. Exp. Sta. Bulletin 215.

and columns if they are looked at as a drawing is usually lettered and read.

Moments at the top of a column, as the column stands in the structure, should be written above the column and those at the bottom of the column, as the column stands in the structure, should be written below the column when the sheet is in position to read the columns. This is necessary because positive moment at the right end of a beam and at the top of a column both represent tendencies to rotate the connected joint in the counterclockwise direction.

It makes no difference whether girder moments are written above or below the girder. Either arrangement may be convenient. Confusion will be avoided by writing column moments parallel to the column and girder moments parallel to the girders.

When any joint is balanced the total moment to the right and to the left of the support is the same, both in absolute value and in sign. The unbalanced moment is the algebraic difference of the moments on the two sides of the joint.

Limitation of Method. From the fact that the terms, "stiffness" and "carry-over factor," have been defined for beams resting on unyielding supports, it follows that direct application of the method is restricted to those cases where the joints do not move during the process of moment distribution. The method, however, can be applied in an indirect way to cases in which the joints are displaced during the moment distribution, as indicated later.

As the method has been stated, it is restricted only by this condition that the joints are not displaced. If this condition is satisfied it makes no difference whether the members are of constant or of varying section, curved or straight, provided the constants (*a*) fixed-end moments at each end, (*b*) stiffness at each end, and (*c*) carry-over factor at each end, are known or can be determined. Such values can be derived by standard methods and may be tabulated for different types of members and conditions of loading.

It will be found that in most cases accuracy is needed only in the fixed-end moments. It does not ordinarily make very much difference how, within reason, the unbalanced moments are distributed, nor, within reason, how much of the distributed moments are carried over.

In the "Illustration" which follows it has been assumed that the members are straight and of uniform section. The stiffnesses, then, are proportional to the moments of inertia, (I), divided by the lengths.

(L), but the relative values given for I/L in this problem might quite as well be the relative stiffness of a series of beams of varying section. In this latter case, however, the carry-over factors for the beams would not be $-\frac{1}{2}$.

Illustration. The illustration given (Fig. 1–1) is entirely academic. It is not intended to represent any particular type of structure nor any probable condition of loading. It has the advantage for the purpose of this paper that it involves all the conditions that can occur in a frame which is made up of straight members and in which the joints are not displaced.

The loads on the frame are supposed to be as indicated. The relative values of I/L for the different members are indicated in circles.

The fixed-end moments in all members are first written. In this problem they are arbitrarily assumed to be as shown, as follows: at A, 0; at B, in BA, 0, and in BC, -100; at C, in CB, -100, in CF, $+80$, in CD, -200, and in CG, -50; at F, $+60$; at G, -50; at D, in DC, -100, and in DE, 0; at E, in ED, 0, and in the cantilever, -10.

Before proceeding to a solution of the problem, attention may be called to the arrangement of the computations. The moments in the girders are written parallel to the girders; those in the columns, parallel to the columns. The original fixed-end moments are written next to the members in which they occur, the successive moments distributed or carried over being written above or below these, but farther from the member.

The arrangement of the moments in the columns in positions above the columns, when the paper is turned into a position to write these moments, for the top of the columns (at B, F, and C), and in positions below the columns for the bottom of the columns (at A, C, and G), is an essential part of the sign convention adopted.

The moment at C in the girder, BC, is written above the girder in order to get it out of the way. Otherwise, it makes no difference whether the moments are written above or below the girder.

The signs of the fixed-end moments are determined by observing the direction of flexure at the ends of the members due to the loads. In order to apply to the columns the ordinary conventions for signs of bending moments it is necessary to turn the drawing of the structure.

The reader should realize that the solution is built up step by step. It is always the last figures showing that are to be operated on — distributed or carried over — so that in ordinary framework there is little chance for confusion as to what step should be taken next.

Distribution. Distribute at each joint the unbalanced moment, as follows:

(1) At A there is no moment (Fig. 1-1).

(2) At B there is an unbalanced moment of -100 on one side of the joint. This moment is distributed to BA and to BC in the ratio, $2:4$, so that the distributed moment to BA is $\frac{2}{2+4}100 = 33.33$ and to BC, $\frac{4}{2+4}100 = 66.67$. The signs are written in the only way possible to balance the joint by giving the same total moment (-33.33) both to left and right of the joint.

(3) At C, the unbalanced moments are, in CB, -100, and in CG, -50, giving a total of -150 on the left of the joint; in CF, $+80$, and in CD, -200, giving a total of -120 on the right of the joint. The total unbalanced moment at the joint, which is the difference between the total moment on the left and on the right of the joint, is 30. This is now distributed in the respective proportions, as follows:

To CB,

$$\frac{4}{4+2+5+1}30 = 10$$

to CF,

$$\frac{2}{4+2+5+1}30 = 5$$

to CD,

$$\frac{5}{4+2+5+1}30 = 12.5$$

and, to CG,

$$\frac{1}{4+2+5+1}30 = 2.5$$

There is only one way to place the signs of the distributed moments so that the total is the same on both sides of the joint. This is done by reducing the excess negative moment on the left and increasing the negative moment on the right.

(4) At F, the unbalanced moment is $+60$. The hinge has no stiffness. The moment, then, is distributed between the member, FC, and the hinge in the ratio, $2:0$; all of it goes to the member. The total balanced moment is $+60 - 60 = 0$, as it must be at a free end.

(5) At G, the abutment is infinitely stiff and the unbalanced moment, -50, is distributed between the member, GC, and the abutment in the ratio, $1:\infty$. The member gets none of it; the end stays fixed.

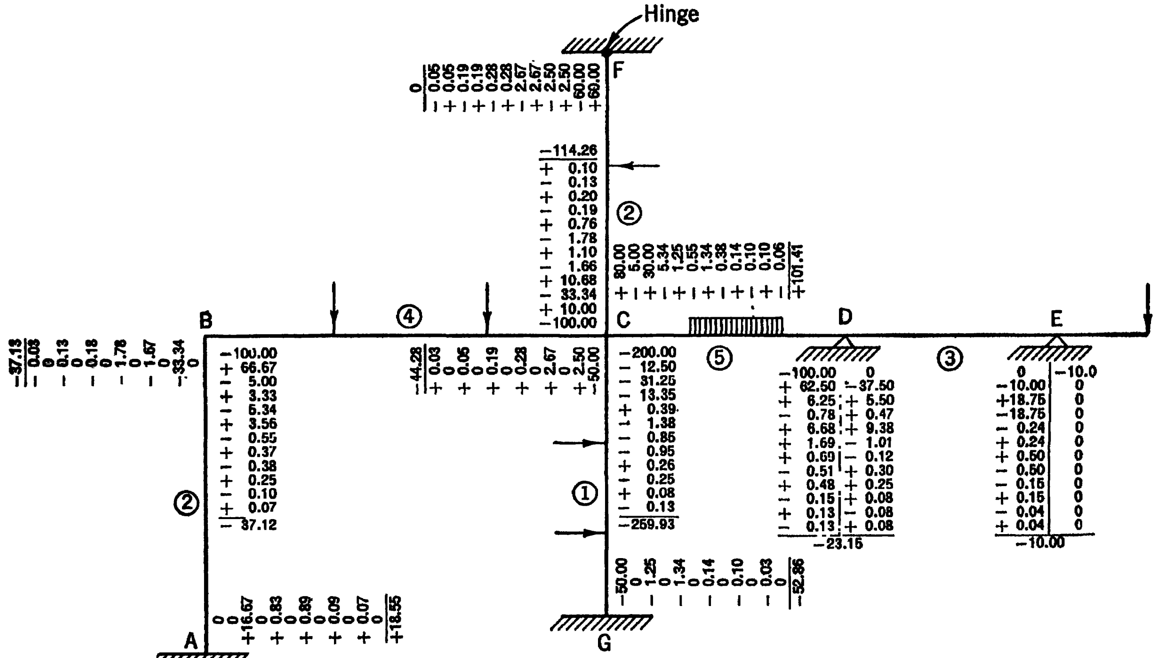

Fig. 1-1. Moment Distribution. Stiffness Factors Are Shown in Circles.

(6) At D, the unbalanced moment, -100, is distributed to DC and to DE in the ratio of $5:3$.

(7) At E, the unbalanced moment is -10 in the cantilever. Since the cantilever has no stiffness, this unbalanced moment is distributed between the beam, ED, and the cantilever in the ratio $3:0$. This means that all of it goes to ED.

All joints have now been balanced. Next, carry over from each end of each member one-half the distributed moment just written, reverse the sign, and write it at the other end of the member. Thus, carry over, successively, in AB, 0 from A to B and $+16.67$ from B to A; in BC, -33.34 from B to C and -5.0 from C to B; in CF, $+2.5$ from C to F and $+30$ from F to C; in CG, 0 from G to C and -1.25 from C to G; in CD, $+6.25$ from C to D and -31.25 from D to C; and in DE, $+18.75$ from D to E and $+5.00$ from E to D.

Distribute the moments just carried over exactly as the original fixed-end moments were distributed. Thus, at A, $+16.67$ is distributed 0 to AB (fixed-ended); at B, -5.0 is distributed as -1.67 and $+3.32$; at C, the unbalanced moment is $(-33.34 + 0) - (+30.00 - 31.25) = -32.09$ which is distributed as $+2.67$, $+10.68$, -5.34, and -13.35; at F, $+2.50$ is distributed as -2.5 to the member; G is fixed-ended; at D, $+1.25$ is distributed as -0.78 and $+0.47$; at E, the unbalanced $+18.75$ is distributed to the member as -18.75.

The moments distributed are now carried over as before and then re-distributed; and the process is repeated as often as desired. The procedure should be stopped after each distribution, however, and a check made to see that statics ($\Sigma M = 0$) is satisfied.

When it is felt that the process has gone far enough, all moments at each end of each member are added to give the total moment at the joint. After the moments at the joints have been determined, all other quantities, such as moments and shears, may be obtained by applying the laws of statics.

Convergence of Results. The distribution herein has been carried out with more precision than is ordinarily necessary, in order to show the convergence of the results. To show the rate of convergence, the successive values of the moments at the joints after successive distributions are given in Table 1–1.

For most purposes the computations might as well have been stopped after the second distribution. Had this been done, the solution would have appeared as shown in Fig. 1–2.

For any practical purpose the computation might in this case have

TABLE 1-1. CONVERGENCE OF RESULTS

Successive values of bending moment at joint		After one distribution (two rows of figures)	After two distributions (four rows of figures)	After three distributions (six rows of figures)	After four distributions (eight rows of figures)	After five distributions (ten rows of figures)	After six distributions (twelve rows of figures)
A	....	0	+ 16.67	+ 17.50	+ 18.39	+ 18.48	+ 18.55
B	....	− 33.34	− 35.01	− 36.79	− 36.97	− 37.10	− 37.13
C	In *CB*	− 90.00	−112.66	−113.22	−114.24	−114.23	−114.26
	" *CF*	+ 75.00	+ 99.66	+100.36	+101.32	+101.36	+101.41
	" *CD*	−212.50	−257.10	−258.09	−259.89	−259.88	−259.93
	" *CG*	− 47.50	− 44.83	− 44.55	− 44.36	− 44.31	− 44.28
D	....	− 37.50	− 32.03	− 23.66	− 23.48	− 23.15	− 23.15
E	....	− 10.00	− 10.00	− 10.00	− 10.00	− 10.00	− 10.00
F	....	0	0	0	0	0	0
G	....	− 50.00	− 51.25	− 52.59	− 52.73	− 52.83	− 52.86

been stopped after the third distribution. In general, two or three distributions are sufficient. This is not true in all instances, but in any case the exactness of the solution at any stage will be indicated by the magnitude of the moments carried over in the members.

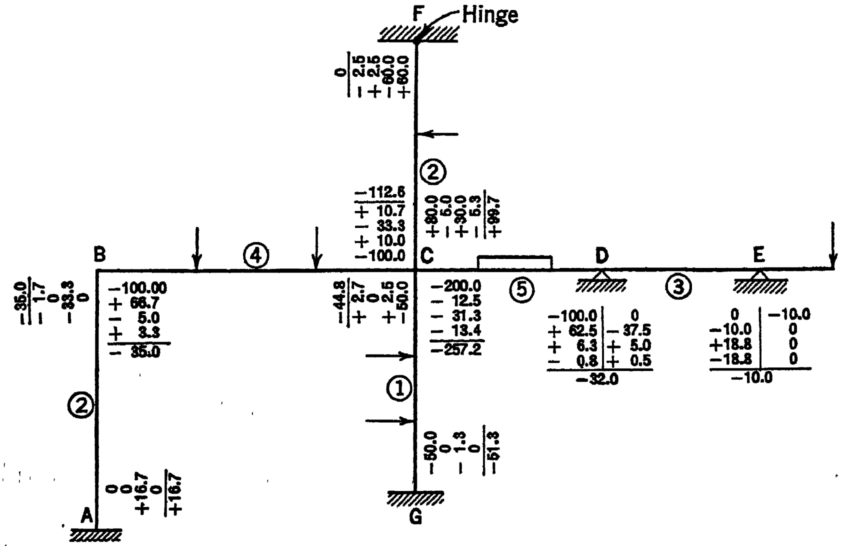

FIG. 1-2. REDUCED NUMBER OF JOINT MOMENT DISTRIBUTIONS.

Variations of the Method. The writer has developed and used at ifferent times several variations of the method shown, but the original ethod is itself so simple and so easy to remember that he finds himself clined to discard the variants.

One variant is perhaps worth recording. It is rather tedious to carry moments out to the end of a member which is free to rotate and then balance the moment and carry it back again. This may be avoided by releasing the free end once for all and leaving it free. In this case, for beams of constant section, the stiffness of the beam is to be taken three-fourths as great * as the relative I/L-value would indicate. After the end of the beam is once released, no moments are carried over to it.

Correcting for Side-Sway

Single square or trapezoidal frames, portals, L-frames, box culverts, and similar structures act as simple continuous beams if there is no transverse deflection. If they are symmetrical as to form and loading, they will not deflect sidewise and if they are restrained against sidewise movement, they cannot so deflect.

Side-sway of frames due to dissymmetry of the frame is rarely an important factor in design. Correction for side-sway may be made by a method which may be applied also in cases of transverse loading on bents. The method is to consider that the bent does not sway sidewise and analyze it as a series of continuous beams. The total shear in the legs will not now, except by accident, equal the shear which is known to exist. The difference must be a force which prevents side-sway.

Now, assume all joints held against rotation, but the top of the bent moved sidewise. Assume any series of fixed-end moments in the legs such that all legs have the same deflection. In this case for members of uniform section fixed-end moments in columns vary as I/L^2. Distribute these fixed-end moments and find the total shear in the legs. The changes in moments due to side-sway will then be to the moments just computed in the same algebraic ratio as the total unbalanced horizontal shear in the legs due to side-sway when the frame is analyzed as a continuous girder is to the shear just computed.

Multi-Storied Bents. Bents of more than one story, subject to sidesway, either as a result of unbalanced loading or due to horizontal forces, may be solved by this method. It is understood that exact solution of such problems is not commonly of great interest. It is the approximate effect that is desired rather than exact analysis.

To analyze by this method a two-story bent it will be necessary to

* The moment needed to produce a given rotation at one end of a beam when the other end is free is three-fourths as great as if the other end is fixed

make two configurations — one for each story. From the assumed shear in each story (producing, of course, shears in the other stories), a set of moment values may be obtained. These may be combined to obtain the true shears, and from the true shears the true moments follow.

General Application of the Method. The method herein indicated of distributing unbalanced moments may be extended to include unbalanced joint forces. As thus extended it has very wide application. Horizontal or vertical reactions may be distributed and carried over and thus a quick estimate made of the effect of many complicating elements in design. The writer has used it in studying such problems as continuous arch series, the effect of the deflection of supporting girders, and other phenomena.

An obvious application of moment distribution occurs in the computation of secondary stresses in trusses. Many other applications will doubtless suggest themselves, but it has been thought best to restrict this paper chiefly to continuous frames in which the joints do not move.

Conclusion. The paper has been confined to a method of analysis, because it has seemed wiser to so restrict it. It is not then an oversight that it does not deal with: (1) Methods of constructing curves of maximum moments; (2) methods of constructing curves of maximum shears; (3) the importance of analyses for continuity in the design of concrete girders; (4) flexural stresses in concrete columns; (5) methods of constructing influence lines; (6) the degree to which continuity exists in ordinary steel frames; (7) continuity in welded steel frames; (8) plastic deformation beyond the yield point as an element in interpreting secondary stress computations; (9) the effect of time yield on moments and shears in continuous concrete frames; (10) plastic flow of concrete as a factor in the design of continuous concrete frames; (11) whether in concrete frames it is better to guess at the moments, to take results from studies made by Winkler fifty years ago, or to compute them; (12) the effect of torsion of connecting members; (13) the relative economy of continuous structures; (14) the relative flexibility of continuous structures; (15) the application of methods of continuous frame analysis to the design of flat slabs; (16) probability of loading and reversal of stress as factors in the design of continuous frames; (17) the relation of precision in the determination of shears and moments to precision in the determination of fiber stresses; and a dozen other considerations bearing on the design of continuous frames.

A method of analysis has value if it is ultimately useful to the designer; not otherwise. There are apparently three schools of thought as to the value of analyses of continuous frames. Some say, "Since these problems cannot be solved with exactness because of physical uncertainties, why try to solve them at all?" Others say, "The values of the moments and shears cannot be found exactly; do not try to find them exactly; use a method of analysis which will combine reasonable precision with speed." Still others say, "It is best to be absolutely exact in the analysis and to introduce all elements of judgment after making the analysis."

The writer belongs to the second school; he respects but finds difficulty in understanding the viewpoint of the other two. Those who agree with his viewpoint will find the method herein explained a useful guide to judgment in design.

Members of the last named school of thought should note that the method here presented is absolutely exact if absolute exactness is desired. It is a method of successive approximations; not an approximate method.

References for Chapter 1

(1) Discussions by 38 contributors to the article reprinted in this chapter, Trans. ASCE **96**, 11–156 (1932).

(2) Hardy Cross and N. D. Morgan, *Continuous Frames of Reinforced Concrete* (Wiley, 1932).

(3) Hardy Cross, *Analysis of Flow in Networks of Conduits or Conductors,* University of Illinois Eng. Exp. Sta. Bulletin 286 (1936).

(4) L. E. Grinter, *Theory of Modern Steel Structures* (Macmillan, 1937 and 1949), Vol. 2 Statically Indeterminate Structures, Chapter on Modern Methods of Analysis of Continuous Frames.

CHAPTER 2

STATISTICAL STATE OF STRESS STUDIED BY GRID ANALYSIS

L. E. GRINTER *

Synopsis. By proportioning an *analogous grid* to reproduce the physical deformation of a solid plate or wall it is possible in a convenient way to attack plane stress problems not readily solved by mathematical methods. To accelerate the convergence of the final numerical series, a procedure of estimating, adjusting, and verifying the main direct stresses and joint movements is found effective. This procedure is termed *strain justification* since the word *justify* includes in its meaning both "to adjust" and "to verify." The final steps include joint force relaxation to provide for ultimate joint translations; but moment distribution, because of its simplicity and convenience, is made the main tool of numerical analysis. Emphasis is placed upon the nature of the statistical state of stress obtained and its relationship to experimentally determined stresses and to P/A.

The Statistical Viewpoint of Elasticity. The mathematical theory of elasticity is based upon the three assumptions of continuity, homogeneity, and isotropy. These hypotheses are never fulfilled exactly by any structural material. For idealized conditions, as checked by photoelasticity, the stresses predicted by the mathematical theory of elasticity seem to be closely approached. For common structural materials, including steel, the stress variation across a section ΔX where uniform stress is predicted by mathematical theory must be more nearly as illustrated by Fig. 2–1. However, strain measurements and interference fringe measurements, being controlled by the summation of stresses across many crystals or fibers, fail to demonstrate the stress distribution of Fig. 2–1.

This irregularity of stress distribution for the woods, concretes, metals and commercial plastics is no doubt due to their lack of homo-

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geneity and isotropy and to inherent internal discontinuities. If steel is taken as typical of the best structural materials, we still observe that a microphotograph shows irregularity of crystalline structure, directional alignment due to rolling, color variation representing change from grain to matrix with undoubted variation of modulus of elasticity, and many small discontinuities. When the subcrystalline structure is considered, discontinuities are spread [illegible] the material, but these seem less significant.

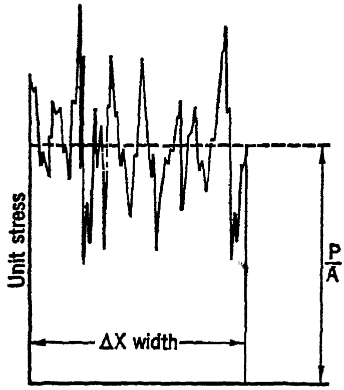


Fig. 2–1. Statistical Meaning of P/A Stress.

The irregularity of stress distribution pictured by Fig. 2–1 leads us to the realization that elastic deformation and ultimately plastic failure are matters controlled by the theory of probability rather than the ideal mathematical relationships dependent upon homogeneity, isotropy, and continuity. As load increases, internal adjustments of a nonelastic nature occur, and these no doubt begin at loads well within the common range of working conditions in a structural member. Initial or fabrication stresses add to the redistribution problem and enhance the importance of a statistical interpretation of computed stresses. Such calculated stresses should, therefore, be taken at their true value which is that of relative numbers representing probability of the extent of the remaining quasi-elastic range before severe permanent deformation can be expected. This is a far more realistic picture of their value in design than the highly idealized concept of maximum stress at a point.

Type of Loading. The viewpoint explained above is presented for static loading. No change in viewpoint seems needed wherever fracture is preceded by measurable ductile deformation. Even cleavage fracture is now accepted to involve observable ductility. Impact fracture may also be ductile. Fatigue fractures should be studied from this point of view for those materials, such as concrete and zinc, which show no measurable reduction of fatigue strength due to stress raisers. If a stress raiser exists which is known to reduce the fatigue strength of the material to be used and if the controlling feature of the design is fatigue resistance, the statistical concept should be adjusted to make due allowance for that fact.

Statistical Interpretation of Grid Forces. Consider a stiff grid of vertical and horizontal members rigidly connected together as a substitute structure for a plate or wall in the vertical plane as shown by Fig. 2–2. Neglecting, for the moment, consideration of the effect of Poisson's ratio and the influence of the cross sections of horizontal and vertical members, it is still clear that the combined forces in a grid member or in the four grid members meeting at a joint can never exactly represent or give rise to the maximum stress computed by the mathematical theory of elasticity at the corresponding point in the plate. This lack of correspondence is due to the physical differences in the two structures: (1) The grid has directional properties, it is not isotropic. It may prove to be more or less nonisotropic than a particular structural material. (2) The grid is certainly not continuous although it provides a geometrical continuity pattern that may be compared with the probable internal continuity patterns of structural materials.

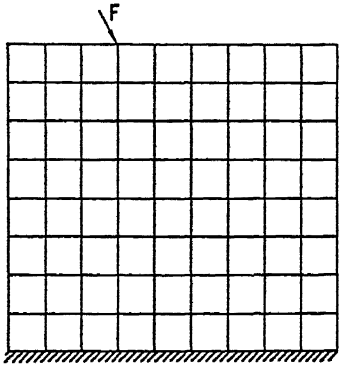


FIG. 2–2. PLANAR GRID.

We conclude then that the physical make-up of the grid gives it a different relationship to the structural plate than that borne by the idealized homogeneous, isotropic, continuous material. In reality, internal grid forces represent statistical or average stresses over lengths controlled by the bar spacing of the grid. It is the author's belief that in many cases such statistical stresses when obtained from a grid of proper unit dimensions or bar spacing will give more useful relative numbers for picturing the remaining quasi-elastic range before structural damage results than will the point stresses determined by the mathematical theory of elasticity. The lattice or unit dimension of the grid for this purpose needs study since *there should clearly be a finer subdivision of the grid where the stress gradient steepens.* This approach to analysis and design might properly be termed the acceptance of a *Statistical Theory of Elasticity.* Although it is particularly compatible with grid analysis, the statistical state of stress is an independent concept useful in interpreting any calculated stress or measured strain. Note again that measured strains and interference fringes give us only the P/A stress in Fig. 2–1. Yet at a macroscopic hole the

STRESSES AND DEFORMATIONS OF THE GRID

Analysis of a Structural Grid. The analysis of the internal forces and moments in a planar structural gridwork follows from the procedures that the author developed for the computation of stresses in the frames of tall office buildings.* The only expected variation is that *direct stress deformations,* which usually have negligible influence upon the moments in a building frame, may be expected to be one of the major factors in the deformations of the grid.

All of the tools of relaxation and distribution that apply effectively to structural frameworks may be of use here. In particular, it is clear that whenever *the coordinates of the deformed positions of the joints* can be guessed at, approximated, or in any way crudely calculated, the grid may be forced into this deformed position with joints fixed against rotation. Then a simple relaxation of joint moment constraints will produce a stress analysis of the same degree of accuracy as the initial estimate of the deformed coordinates. Such an analysis can, of course, be improved by successive corrections to produce any desired accuracy. This procedure has long been accepted as a standard method of computing wind stresses in tall buildings.

Since the procedure of grid analysis is largely solved, we will proceed to study the problem of proportioning a gridwork to replace the over-all elastic properties of a plate. We may feel confident that the resulting grid can be analyzed without undue difficulty. After some experience we should find ways of reducing the work of grid analysis just as the *simplified method* ** was developed to remove a major part of the labor from the wind-stress analysis of tall buildings. A contribution to such simplification will be found in the author's procedure of "strain justification" for obtaining compatibility between the direct stresses and shears and the corresponding strains in two directions.

Reproducing Plate Deformations in the Grid. There are three distinct deformation phenomena of a wall or plate that must be reproduced by the gridwork:

(1) Direct deformation must occur as controlled by E.
(2) Lateral deformation must follow as defined by Poisson's ratio, μ.
(3) Shearing deformation (internal rotation) controlled by G must be simulated.

* L. E. Grinter, "Wind stress analysis simplified," Trans. ASCE, 99 (1934), pp. 610–634. Also, *Theory of Modern Steel Structures* (Macmillan, 2d ed., 1949), Vol. 2, pp. 135–148.
** *Ibid.*

We will produce similar structural action in the grid in the following manner:

(1) Columns and beams will shorten and lengthen under their direct stresses.
(2) When a column shortens adjacent beams will be lengthened by μ times the column shortening and *vice versa.*
(3) Joints will be permitted to rotate and members will bend.
(4) If it is found that the influence of μ and of $G = \dfrac{E}{2(1 + \mu)}$ can be represented accurately by steps (2) and (3), shearing deformation of the members of the gridwork will be omitted from consideration.

Since the gridwork is a hypothetical structure even though treated in a physical way, we are free to postulate its properties to suit our convenience. The relative deflection of the ends of a beam due to shear can be made up by an equal amount of bending deflection (with slight change in the depth of the member). Hence, it will be feasible to work entirely with bending and direct-stress deformations.

Proportions of the Gridwork

Direct-Stress Deformations. The deformation of the plate under a uniaxial stress P/A is PL/AE over the length L. See Fig. 2–3. Since

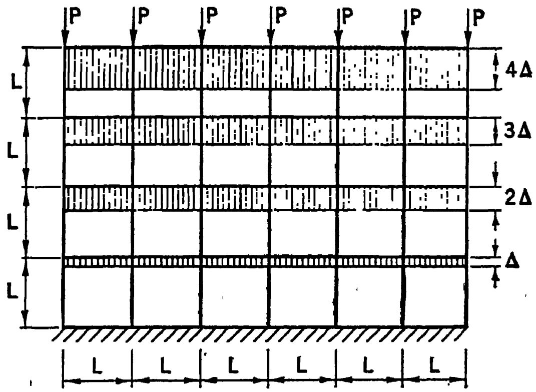


Fig. 2–3. Direct-Stress Deformation of a Grid.

the area of the plate for a width L is Lt, this area must be replaced by each vertical member of the grid when the bar spacing of the grid is L.

$$A = Lt = bd \tag{2-1}$$

where b and d are cross-sectional dimensions of a grid member.

For members of this area the longitudinal deformation of the grid under uniaxial stress corresponds exactly with the longitudinal deformation of the plate.

In equation (2–1), d is the depth of a bar in the plane of the plate and b is its dimension perpendicular to the plate so the $I_{\text{bar}} = \frac{1}{12} bd^3$ for flexure in the plane of the plate.

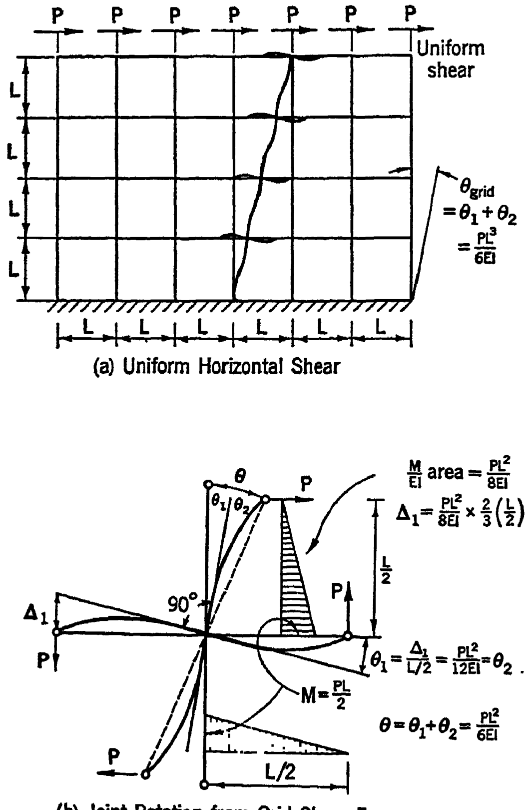


FIG. 2–4. PURE SHEARING DEFORMATION OF A GRID.

Reproducing the Shearing Deformation of the Plate. Consider the influence of a uniform shearing stress, P/A, across the top of the plate or of the grid shown in Fig. 2–4. For the elastic plate

$$G = \frac{s_s}{\theta} = \frac{P/A}{\theta} \tag{2–2}$$

Also

$$G = \frac{E}{2(1 + \mu)} = \frac{P/A}{\theta} \tag{2–3}$$

Hence

$$\theta_{\text{plate}} = \frac{2P(1+\mu)}{AE} \tag{2-4}$$

And from Fig. 2-4 (*b*)

$$\theta_{\text{grid}} = \frac{PL^2}{6EI} \tag{2-5}$$

By equating the θ values of the plate and the grid we obtain

$$\frac{2P(1+\mu)}{AE} = \frac{PL^2}{6EI} \tag{2-6}$$

Substituting $A = Lt = bd$ and $I = \frac{1}{12}bd^3$, and canceling terms, we have

$$\frac{L^2}{d^2} = (1+\mu) \quad \text{or} \quad \frac{d}{L} = \frac{1}{\sqrt{1+\mu}} = \sqrt{1-\mu+\mu^2-\mu^3+\cdots} \tag{2-7}$$

For convenience, we may write

$$d = \frac{L}{\sqrt{1+\mu}} \quad \text{and} \quad b = \frac{Lt}{d} = t\sqrt{1+\mu} \tag{2-8}$$

When $\mu = 0$,

$$d = L \quad \text{and} \quad b = t$$

When $\mu = 0.25$,

$$d = 0.89L \quad \text{and} \quad b = 1.12t$$

Member Sizes. Hence, the member sizes of the grid are established. The area bd of each member replaces the corresponding area Lt of the plate and the ratio b/d is the ratio (t/L) times the quantity $(1+\mu)$. It is to be noted that the grid members in the horizontal direction are identical with those in the vertical direction. It may at first seem peculiar that the grid members in either direction replace the entire area of the plate. However, when we realize that plate material does have the ability to resist forces in both directions while grid members are uniaxial, we find this relationship reasonable.

Division of Load between Columns and Beams. Although their positions may reverse with change in direction of the main loads, we will temporarily find it convenient to look upon vertical members as columns and horizontal members as beams or girders.

Length of column $= L$

Breadth of column = depth of beam $= d = \dfrac{L}{\sqrt{1+\mu}}$

Area of cross-section of column or beam $= bd = t\sqrt{1+\mu}\left(\frac{L}{\sqrt{1+\mu}}\right) = tL$

Force necessary to shorten column one unit. See Fig. 2–5.

$$\Delta = 1.0 = \frac{PL}{AE}$$

Hence,

$$P = \frac{AE}{L} = \frac{tLE}{L} = tE$$

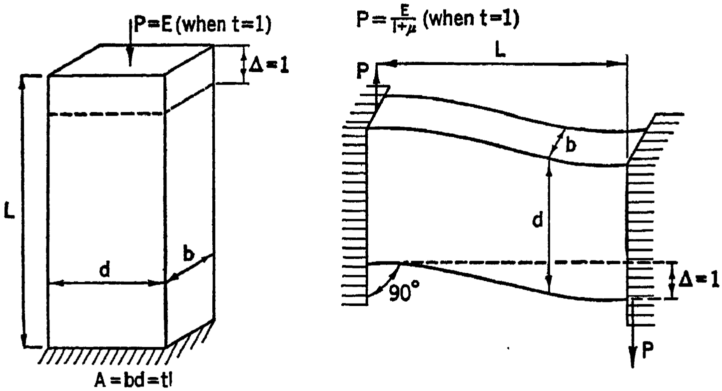


FIG. 2–5. UNIT VERTICAL DEFORMATIONS OF COLUMN AND BEAM.

For unit plate thickness, $t = 1$ and therefore

$$P_{col} = E \text{ (for unit deformation and unit plate thickness)} \qquad (2\text{–}9)$$

Force necessary to deflect beam one unit by flexure alone as a fixed-end beam or as a double cantilever. See Fig. 2–5.

$$\Delta = 1.0 = 2\left(\frac{P(L/2)^3}{3EI}\right) = \frac{PL^3}{12EI}$$

But

$$I = (\tfrac{1}{12})bd^3 = \left(\frac{t\sqrt{1+\mu}}{12}\right)\left(\frac{L}{\sqrt{1+\mu}}\right)^3 = \frac{tL^3}{12(1+\mu)}$$

Hence,

$$\Delta = 1.0 = \frac{PL^3}{12E}\left(\frac{12(1+\mu)}{tL^3}\right) = \frac{P(1+\mu)}{tE}$$

For unit plate thickness, $t = 1$ and therefore

$$P_{beam} = \frac{E}{1+\mu} \text{ (for unit deflection and unit plate thickness)} \qquad (2\text{–}10)$$

Relative Stiffness for Force Distribution.

Column stiffness $= 1.0$ (from $P = E$ when $\Delta = 1$ and $t = 1$) (2–11)

$$\text{Girder stiffness} = \frac{1.0}{1+\mu}\left(\text{from } P = \frac{E}{1+\mu} \text{ when } \Delta = 1 \text{ and } t = 1\right) \tag{2–12}$$

These relations control the division of load or of joint force between the columns and girders meeting at any joint.

When $\mu = 0$, $d = L$, $b = t$ and the force distribution to columns and girders meeting at a joint is the same.

Relative Stiffness for Moment Distribution. Since the cross sections of columns and girders are identical, the resistances to joint rotation are the same for all members.

Secondary Influences in Plate and Grid

Deformations for Loads at 45 Degrees to Beams and Columns. The extent of the influence of the non-isotropic properties of the grid can be checked by studying the distortions due to loads at 45 degrees to the directions of the beams and columns. The necessary calculations including the influence of Poisson's ratio are presented in an appendix. It is sufficient here to point out that the deformation in line with a load at 45 degrees to the direction of the columns is found to be the same as the deformation when the load is placed on the grid in line with the columns. Also, if the procedure of lengthening beams by Poisson's ratio times the shortening of columns is carried out for loads at 45 degrees to the members of the grid, we find that a compressive deformation of unity along a line at 45 degrees to the directions of the members gives rise to an increase in the grid dimensions by the factor 1.0μ at 90 degrees thereto. Hence, the effect of Poisson's ratio upon deformations is introduced in the diagonal directions as well as the coordinate directions. We conclude, therefore, that the directional geometry of the grid does not disqualify it as an analytical substitute for the structural plate.

The Introduction of Poisson's Ratio. Since the main example to be solved will not be complicated by the introduction of Poisson's ratio, it is necessary to explain how this ratio may be introduced into grid analysis when necessary. *Any value of Poisson's ratio may be introduced into this numerical procedure.* The mathematical theory of elasticity has shown, however, that the influence of Poisson's ratio in plane stress problems is usually negligible if not entirely zero. The

class of problems where μ does not enter into the equations for internal stresses are those of plates loaded around a continuous free boundary. Body forces and impressed distortions such as shrinkage effects give rise to stresses that are influenced by the value of Poisson's ratio, at least to some degree. However, deformations always depend upon Poisson's ratio.

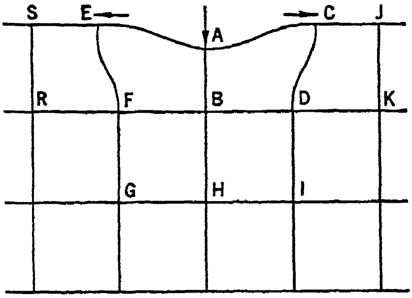


FIG. 2–6. EFFECT OF POISSON'S RATIO.

In Fig. 2–6 we have a gridwork shown with a vertical load at A. As a first step this load is permitted to compress the vertical member AB and deflect the horizontal members EA and AC. With the shortening of AB by Δ_{AB} there is a lateral deformation developed which is expressed as a lengthening of the horizontal members EA and AC by $0.5\mu\,\Delta_{AB}$ if this lateral movement is unresisted. However, if the joints E and C are not permitted to move horizontally, the result is to produce a stress $0.5\mu S_{AB}$ in the two members EA and AC. Similarly, if there is a stress introduced into BH as well as AB, there will be lateral stresses of $0.5\mu\left(\frac{S_{AB} + S_{BH}}{2}\right)$ introduced into FB and BD.

Evidently, as shown in Fig. 2–6, the stress $0.5\mu S_{AB}$ in the members EA and AC produce joint forces acting outward at E and C. These joints must be permitted to translate laterally until equilibrium is reached. As E translates to the left the average of the stresses in AE and ES give rise to a vertical stress in EF, producing an unbalanced vertical force at F.

$$S_{EF} = 0.5\mu\left(\frac{S_{EA} + S_{ES}}{2}\right) = (0.5\mu)^2\left(\frac{0.67S_{AB} + 0.33S_{AB}}{2}\right) \text{ approximately.} \tag{2–13}$$

Hence, $S_{EF} = \frac{\mu^2}{8} S_{AB}$ approximately, which is quite negligible. This result leads us to conclude that *the influence of Poisson's ratio need not be introduced with great precision.* Actually, it may be introduced normally into the force distribution procedure without greatly increasing the work involved. In some cases the influence of Poisson's ratio can be introduced more conveniently as a final numerical correction. As has been mentioned, its effect on plane stresses is often either zero or nearly so.

Accelerating the Convergence

Successive Relaxation of Joints Proves Tedious. If a boundary force exists at the joint D as is shown in Fig. 2–7, the first step might be to permit the joint D to move vertically while the adjacent joints C, K and E were held fixed against translation and rotation. The force distribution between the three members meeting at D would be determined from the relative stiffness controlled by equations (2–11) and (2–12). Each of the distributed forces in DC, DE and DK would be carried over to the joints C, E and K without reduction. In sequence, the joints C, E and K could then be released for vertical translation, and so on, until the original force at D had been distributed by repeated vertical translations to all joints of the grid without permitting joint rotations.

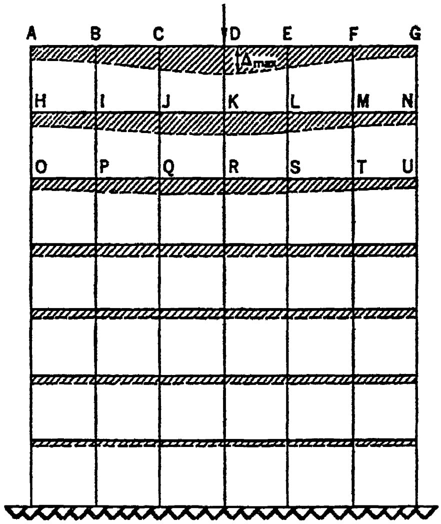


Fig. 2–7. Initial Relative Movements of a Group of Joints.

Sequence of Steps Involved in Successive Relaxation of Joints. To outline the entire procedure as a sequence of steps we have:

(1) Release joints for successive vertical translation under the action of the load until equilibrium is achieved.
(2) Release joints for successive rotation caused by joint moments accompanying the fixed-end shears in horizontal members developed in step (1).
(3) From step (2) there are changes in the shears in all horizontal members, and shears are also introduced into vertical members. Compute these.
(4) On separate sheets permit vertical and horizontal translations of the joints to attain equilibrium under the unbalanced joint forces developed by the shear changes of step (3).
(5) New shears introduced in step (4) produce new joint moments to be balanced and distributed.
(6) The procedure is completed by repeating steps (2) to (5) as often as necessary to reduce the remaining joint forces to negligible

values. Steps (1) to (5) may, of course, be used in a changed order.

Unfortunately the procedure outlined, although simple and direct, is usually so tedious that it cannot be used without modification. In particular, the first distribution of a boundary force throughout the grid by step (1) is unacceptably tedious. The remaining steps converge much more rapidly. Step (1) must therefore be accelerated.

Group Movements of Joints. In order to accelerate convergence of joint translations several devices have been employed:

(*a*) *Block Movements.* When a number of connected joints have joint forces acting in the same direction, even though of different magnitudes, a "block movement" may prove useful. A "block movement" is here defined as translation of several joints without relative motion. Hence, the action corresponds to the movement of a *rigid block.* This device is of limited usefulness.

(*b*) *Relative Movements.* As a starting point in an analysis or at any stage when a reasonable estimate of final joint movements can be made (see Fig. 2–7) we can translate the joints into the pattern of estimated deflections without permitting joint rotations and determine without difficulty the corresponding direct stresses and fixed-end shears in all members. Then a simple distribution of moments will produce joint equilibrium for forces and moments if the assumed joint translations were exactly correct. Any unbalanced joint forces or joint restraints represent the inaccuracy of the estimated distortions. Such restraints may be removed by following steps (4), (5) and (6) of the preceding section. This procedure as illustrated by Fig. 2–7 involves *relative movements* rather than "a block movement" of joints.

Strain Justification

(*c*) *Use of Estimated Direct Stresses.* Since we are dealing with forces it is usually simpler to approximate the direct stresses in the members, at least in one direction, than to estimate joint movements. Such requirements as that of statics in Fig. 2–7 (the sum of all column stresses at any level must equal the load P) may be used effectively to control an estimate of the column direct stresses. Then *by working upward from the fixed base* in Fig. 2–7 these estimated direct stresses may be used to determine an initial vertical translation of each joint with the corresponding fixed-end shears in each horizontal member. From here on, we could make use of the distribution and relaxation

steps (2) to (5) of the preceding section. It is clear that either a fixed base or a symmetrical center line are helpful since, otherwise, there is an additional approximation involved when joint movements are computed relative to a line that may warp under the action of the loads. In such instances *an estimate of the deflections of the base line* has been employed successfully as will be demonstrated in Appendix B by the analysis of a notched beam.

(*d*) *Static Check after Moment Distribution.* In addition to starting an analysis with a group of estimated stresses that (1) obey the requirements of statics and (2) that may be used to compute "relative movements" of the joints, it is possible (3) to *improve the first estimate through rapid or even crude use of moment distribution.* Since force distribution is far slower than moment distribution, it is desirable to reduce the required steps of force distribution as far as possible. After an estimate of say the vertical forces and a calculation of the corresponding fixed-end shears and moments in the horizontal members of the grid, joint moments may be balanced and distributed. Even though this process is carried out with less than two-figure accuracy it will provide an improved set of shears in the horizontal members that should combine with the direct stresses in vertical members to approach joint equilibrium in the vertical direction. An application of statics ($\Sigma V = 0$) to each joint provides us with the restraining forces at the joints. *If these restraining forces are small at most joints, but relatively large at one or more joints, we can use this information to revise the initial estimate of direct stresses and repeat the procedure until equilibrium is more nearly established.* The process of revising the direct stresses in one or both directions so that the corresponding strains and flexural deformations will result in balanced or compatible joint forces is here called "strain justification." The word "justification" includes the necessary dual meaning of adjustment and verification.

(*e*) *Final Steps of Convergence.* When a good estimate of the direct stresses in the major direction is obtained, the completion of the analysis by the repetition of steps (2) to (5) from the preceding section will not prove tedious.

Résumé. Procedure (*d*) offers a device for minimizing the tedious process of distributing joint forces. *Instead, we repeatedly improve an initial estimate of the direct stresses in one or both directions from the information obtained through a relatively crude distribution of moments.* This technique of strain justification offers the added advantage that

elaborate sheets of recorded distributions are avoided. At any stage of the calculations all previous work may be consigned to the wastebasket and only the best available approximation of the direct stresses need be saved. *If this approximation of the direct stresses is essentially exact, a simple balancing of the corresponding fixed-end moments will show that the joints are in both force and moment equilibrium.* Any lack of force equilibrium is then expressed as a series of joint restraints to be eliminated (*a*) by an improved estimate or adjustment of the direct stresses or (*b*) by using steps (3) to (5) of the preceding section.

Example of Shrinkage Stresses

Cooling of a Lineal Weld in a Free Plate. As an example of the application of the procedures outlined in (*d*) and (*e*) of the preceding section, it was decided to present the calculation sheets for determining the stresses in a plate with free boundaries when shrinkage occurs along one center line. The illustration Fig. 2–8 shows the corresponding grid for which a free shrinkage deformation equivalent to the strain accompanying a stress of 27,500 lb per sq in. would have occurred in the members *IH*, *HA*, *AH′* and *H′I′* if they had not been restrained by the remainder of the grid. Hence, these members must be in tension and adjacent members such as *GB* must be in compression.

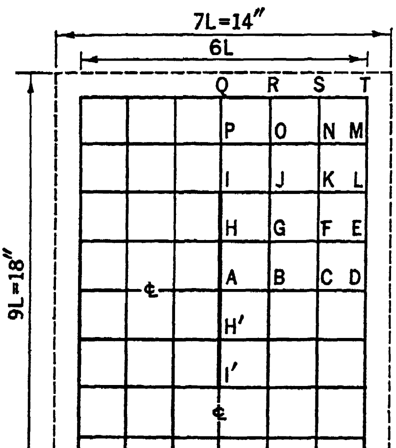


Fig. 2–8. Relationship of Grid to Plate.

In this example, shrinkage is considered to take place in one direction only. In Fig. 2–8 the grid bar that shrinks (*IHAH′I′*) is eight inches long and two inches wide, representing one-seventh of the plate width. Hence, the lateral shrinkage is comparatively unimportant and is neglected although it could have been included in the analysis. If the plate thickness is one-half inch, each bar has an area of $2 \times \frac{1}{2} = 1.0$ sq in. The grid of seven by nine members has the same cross-sectional area in each direction as the $14 \times 18 \times \frac{1}{2}$-in. plate.

The stresses in the horizontal direction are clearly not very important. However, they will be analyzed as far as necessary to determine

their influence upon the vertical stresses. In this instance Poisson's ratio will influence stresses in both directions but the effect is small. In order that the reader may find it possible actually to follow the numerical procedure from sheet to sheet without undue confusion, the value of Poisson's ratio is taken to be zero. For a value of $\mu \neq 0$

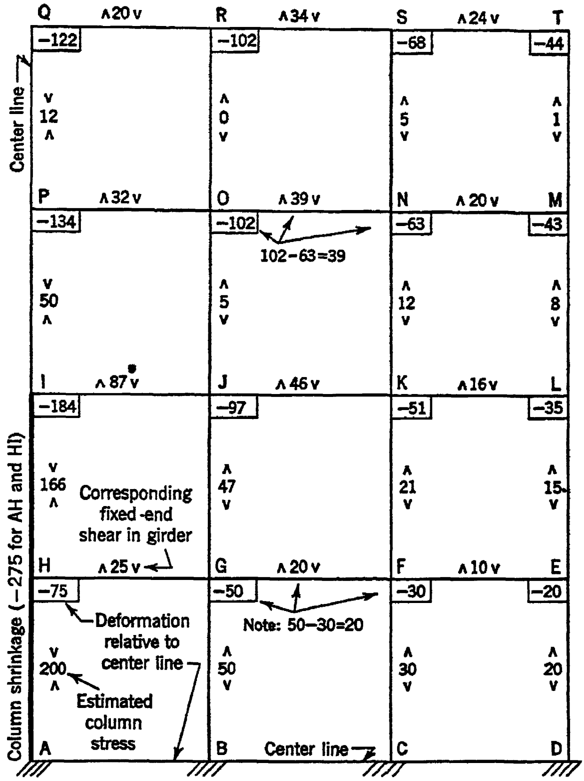


FIG. 2-9. ESTIMATE OF COLUMN STRESSES.

its influence could be introduced at an early stage of the numerical procedure or be added as a final correction.

Numerical Correction Sheets. By the procedure to be used there are no calculations other than the simplest mental arithmetic. The only serious problem involved is to record the numerical corrections in as nearly a foolproof manner as possible. Long experience has shown the system of Fig. 2–10 to be the most effective for moment distribution. Signs of moments indicate *the action of the moment in the member on the joint,* negative moment being counterclockwise. The unbalanced moment divides equally between the members meeting at a joint and the carry-over factor is +0.5. This sign convention is

different from that used in Chapter 1.* A single line is drawn below the moments in all the members meeting at a joint when it has just been balanced. A double line is used above a summation.

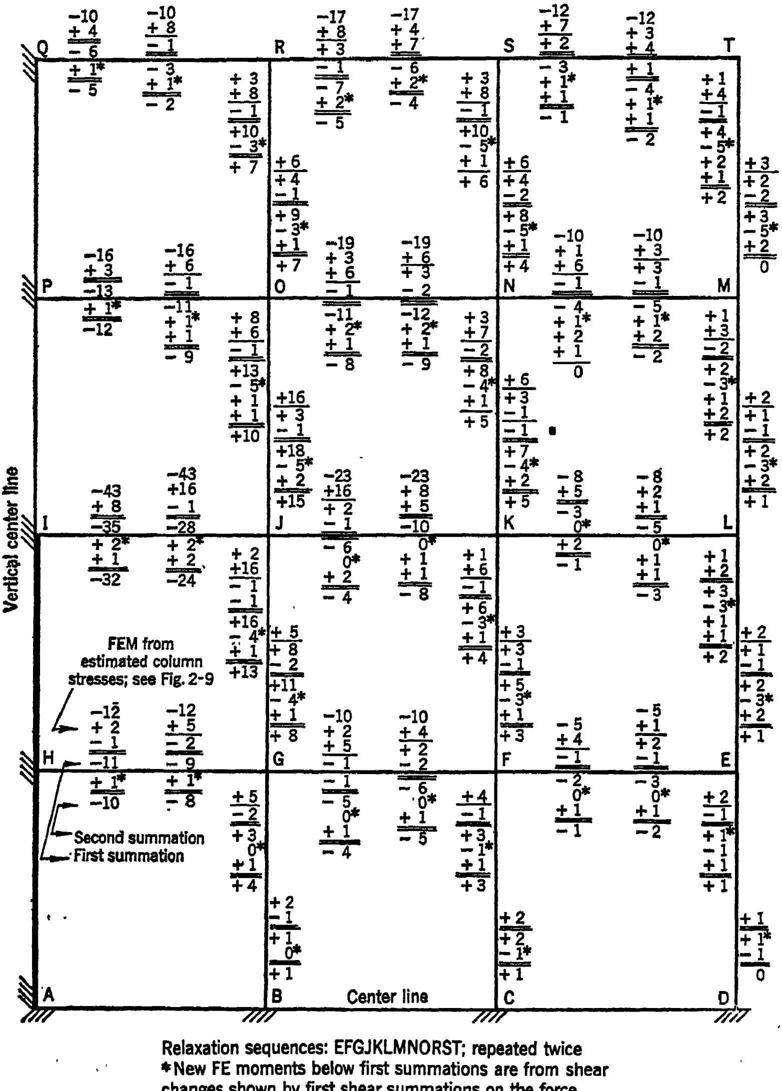
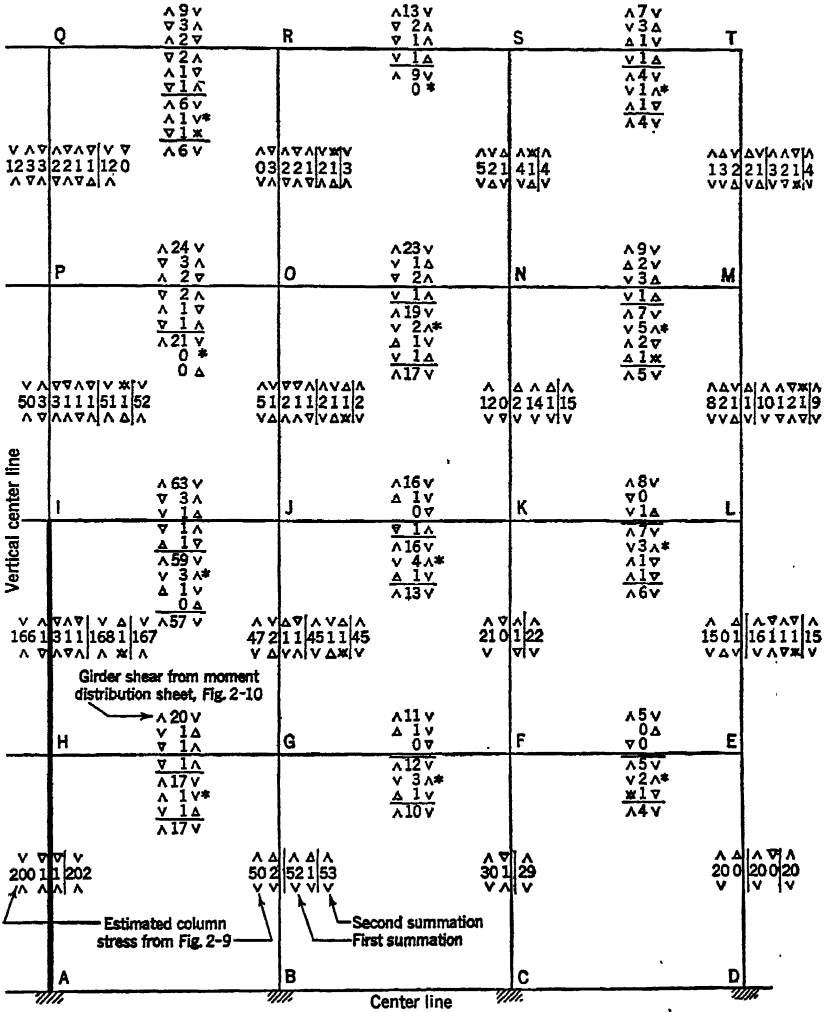


FIG. 2-10. DISTRIBUTION OF FIXED-END MOMENTS.

* Considerable difference will be found between the techniques of moment distribution used in Fig. 2-10 and Fig. 1-1 or 1-2. However, the basic procedures are the same. The techniques followed will be explained later. For a detailed explanation, see reference 7 at the end of this chapter.

For force distribution the system shown on Fig. 2–11 is used. One vertical column of numbers represents the changing shear in a horizontal member and one horizontal line of numbers represents the chang-

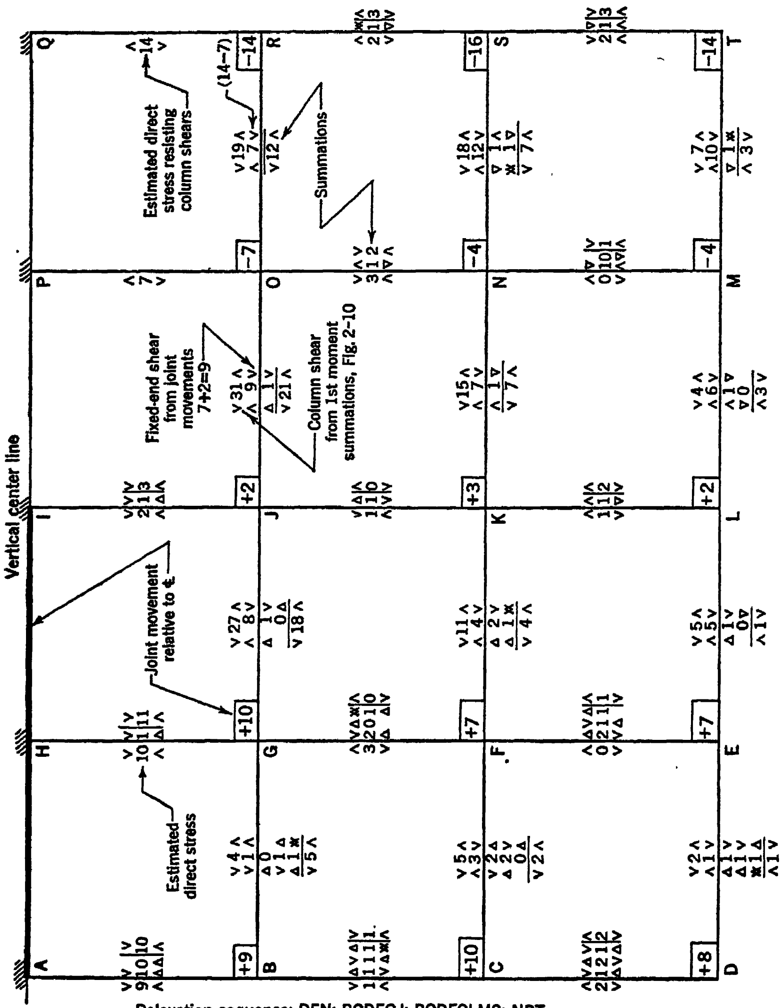


Relaxation sequence: EGHIJKNPQRT; MOPQRS; FHILOPQT; JMQ
Second sequence: GILMQ; EJLMOR; EFGHJNPRS
*Correction shears to make total shears agree with second summations of moments on Fig. 2–10

FIG. 2–11. DISTRIBUTION OF V-FORCES.

ing direct stress in a vertical member. The arrow heads show *the action of the force in the member on the adjacent joints,* the two arrow heads in each case being oppositely directed. Here a single line precedes a summation. Note that whenever a joint restraint is released,

a stress or shear enters each of the members intersecting at that joint. If the joint restraint cannot be divided into equal parts (when $\mu = 0$) corresponding to these three or four intersecting members, the numbers



Relaxation sequence: DFN; BCDEGJ; BCDFGLMS; NRT
*Correction shears could be obtained from moment distribution sheet for agreement with second moment summations, but these were neglected since only V-forces were desired

Fig. 2–12. Distribution of *H*-Forces.

recorded are the closest approximation possible, which may even require that a zero correction stress or shear be assigned to one or more members. Whatever the correction may be, its direction is indicated by an arrow placed nearest to the joint just balanced and the opposing

arrow on the far side is added to show its influence upon the adjacent joint at the opposite end of the member.

Open and Closed Arrow Heads. It will be noticed that some arrow heads are open and some are closed. This simple device along with the record of the relaxation sequence at the bottom of the sheet makes it possible for the operator to recheck his figures step by step. A closed arrow head is used *only* to indicate a force introduced at the nearest joint as a balancing force. The directions of the original forces in the members and the carry-over forces are indicated by open arrow heads. Toward the end of the convergence the cancelled arrow head XX provides a termination of the carry-over procedure. These devices make it possible for us to record single lines or columns of numerals on the force distribution sheets while double columns are necessary on the moment distribution sheet where the carry-over factor is not unity.

Original Estimate of Vertical Column Stresses. On Fig. 2–9 is shown the estimate of stresses in the vertical members, here called columns. If given a free opportunity for shrinkage, the members AH and HI would shorten by an amount equivalent to the deformation due to a stress of 27,500 lb per sq in. The restraint of the grid places these members in tension, which is estimated to be 20,000 lb per sq in. for AH (recorded as 200 on Fig. 2–9) and 16,600 lb per sq in. in HI which is recorded as 166. The member AH is the central member of the grid. Therefore its recorded stress of $+200$ must be balanced by -100 on each side in three columns. Hence, we find -50 in BG, -30 in CF, and -20 in DE, a total of -100 to the right of AH. Since all members have the same cross-sectional areas which are here considered to be unity, the unit stresses will also serve as *total stresses* or as forces in the members.

In a similar manner all vertical column stresses were estimated and the resulting joint movements, recorded in boxes at the joints of Fig. 2–9, were obtained. Here it was found convenient to record just the value of the summation of the direct stresses for all members above the base or center line. A direct stress must of course be divided by AE and multiplied by L to become a displacement. But then the relative displacements of two joints such as G and F would have to be multiplied by AE and divided by L to become the shearing force in the member GF. This manipulation was avoided in Fig. 2–9 by recording the summation of the direct stresses as relative joint displacements. This relationship would be changed slightly if $\mu \neq 0$, as is clarified by Fig. 2–5 and equations (2–9) and (2–10).

Revised Estimates of Vertical Column Stresses. Actually the column direct stresses recorded on Fig. 2–9 are not the first set of estimated direct stresses. The first estimate was so crude that upon determination of the fixed-end shears in the horizontal girders it was clear that the equation of statics $\Sigma V = 0$ was very far from being satisfied at a number of joints. Since a revision of the column direct stresses would entail very little additional work it was immediately decided upon.

The second estimate of column direct stresses proved to be better than the first since the fixed-end girder shears, when calculated, appeared reasonable. Hence, the next step of balancing joint moments was carried out. Then the resulting girder shears along with the estimated direct stresses were tested at each joint by use of the equation of statics, $\Sigma V = 0$. In fact, the corresponding joint restraint in the V-direction was determined at each joint. *A picture of the grid was then drawn with a load at each joint of equal and opposite value to the unwanted joint restraint.* This procedure of revising direct stresses and joint movements has previously been described under the heading of "strain justification" for compatibility.

Final Estimate of Vertical Column Stresses. Since the final stresses in the vertical columns would be those of the second estimate plus or minus those produced by the set of joint forces that are equal and opposite to the joint restraints, it became possible and desirable to make a third estimate of the column direct stresses. This is the estimate shown in Fig. 2–9. Since this is merely an estimate, the smaller effect of joint restraints in the lateral or horizontal direction upon vertical column stresses was neglected. This influence was too subtle to be introduced into the estimate, but it will be obtained through the balancing and distribution of forces and moments in Figs. 2–10, 2–11, and 2–12. However, it will be found that every minute spent in improving the original estimate of direct stresses is likely to save several minutes of time otherwise consumed in repeated relaxation of joint restraints. The procedure of strain justification is an important time saver.

Final Outline of Numerical Procedure

Detailed Steps in the Removal of Restraints. We understand that the procedure is initiated by estimating the direct stresses in one or both directions in the grid. If the stresses in the vertical direction as in Fig. 2–9 are clearly the controlling stresses, they may be given first

consideration. Hence, we have for this problem the following steps of analysis:

(1) Estimate the vertical or column direct stresses.
(2) Determine the corresponding joint movements (vertically) relative to the fixed horizontal center line.
(3) Compute the shears in the horizontal girders caused by these joint movements.
(4) Check the forces at each joint very roughly for agreement with statics by use of the equation, $\Sigma V = 0$. Anticipate some reduction in the shears in horizontal girders due to moment distribution.
(5) Revise steps (1), (2), (3) until step (4) is satisfied.
(6) Balance and distribute fixed-end girder moments into all members of the grid. This step changes the girder shears and introduces shears into the columns.
(7) Use the revised girder shears from (6) along with the last estimate of column direct stresses to again check the joints for force equilibrium by use of the equation, $\Sigma V = 0$ at each joint. If seriously unbalanced joints are found, revise the estimate of vertical column direct stresses, step (1), and repeat steps (2), (3) and (6).
(8) Permit joints to translate vertically under column direct stresses and girder shears. A joint by joint relaxation sequence is satisfactory here.
(9) Compute horizontal joint restraints from column shears, step (6), and estimate the direct stresses in the girders that will occur when these restraints are removed.
(10) Compute joint movements in the H-direction relative to the fixed vertical center line due to the estimated direct stresses in the girders from step (9).
(11) Introduce fixed-end column shears due to these relative joint movements along with column shears from step (6), moment distribution.
(12) Permit joints to translate horizontally under girder direct stresses, step (9), and column shears, steps (6) and (11).
(13) Steps (8) and (12) have disrupted the moment balance of step (6). Hence, correction shears from steps (8) and (12) are introduced as fixed-end moments on the moment distribution sheet and joint moments are rebalanced and distributed.
(14) Corrections to the girder and column shears from step (13) are introduced on the H and V force distribution sheets, and joints

are again permitted to translate to attain internal equilibrium, i.e., $\Sigma H = 0$ and $\Sigma V = 0$ at each joint. If a reasonable estimate of direct stresses was obtained through steps (1) to (5) and step (9), the changes in the shears will probably be too small to influence appreciably the balance of joint moments. Otherwise steps (13) and (14) are to be repeated.

(15) In the example of Figs. 2–9 to 2–12, steps (1) to (13) were completed but in step (14) the correction shears were introduced only on the V-force distribution sheet, Fig. 2–11. The H-forces were found to be small and the influence of introducing corrections on the H-force distribution sheet, Fig. 2–12, would not have influenced the V-forces which were desired.

Recording the Numerical Steps of Convergence

Sheets of Numerical Calculations. After the estimate of the direct stresses and the calculation of the corresponding relative joint movements and girder shears in Fig. 2–9, the first numerical calculation sheet is the balancing and distribution of moments in Fig. 2–10. The fixed-end moments in any girder are taken to be one-half of the fixed-end girder shear for that member from Fig. 2–9. In member HG the shear $\wedge 25 \vee$ in Fig. 2–9 gives rise to fixed-end moments of -12 for HG in Fig. 2–10. Thus, we are assuming a member length L of unity. The sum of the unbalanced moments at a joint is divided equally between the intersecting members. A moment that tends to rotate the adjacent joint clockwise is given a plus sign. Carry-over moments are of the same sign as the balancing moments that produce them.

When columns of moments are summed, the sum of the two end moments for any member equals the final shear in that member. The first summations of balanced moments from Fig. 2–10 (numbers below first double lines =), give rise to the first girder shears in Fig. 2–11 and the first column shears in Fig. 2–12. For example, in member HG of Fig. 2–10 the first moment summations -11 and -9 give rise to the shear $\wedge 20 \vee$ in HG on Fig. 2–11. Similarly, in the member JG of Fig. 2–10 the moment summations $+16$ and $+11$ give rise to the column shear $\vee 27 \wedge$ on Fig. 2–12.

Force Distribution or Joint Translations. When a joint restraint is relaxed and translation is permitted, forces adjust themselves to produce equilibrium. Consider joint E which is the first joint to be balanced on Fig. 2–11. There is a compression of 15 acting downward from LE, a shear of 5 acting downward from FE and a com-

pression of 20 acting upward from *ED*. Hence, this joint is already balanced and $\Sigma V = 0$. The balancing forces are zero and are written as ∆0 for each member. At *G*, however, we have 47 acting down from *GJ*, 50 acting up from *BG*, 20 acting down from *HG* and 11 acting up from *GF*. The resultant is 6 acting down which must be balanced by four upward correction forces. Each, of course, should be 1.5, but to avoid fractions an upward force of 2 is added to each of the two columns and an upward force of 1 is added to each girder at *G*.

On Fig. 2–12 we have for the *H*-forces the equivalent of Figs. 2–9 and 2–11 for the *V*-forces. Estimated stresses are given for the horizontal members with the corresponding joint translations relative to the vertical center line. These *relative joint movements* give rise to column shears such as ∧7∨, the second recorded number for the member *RO*. The first number ∨19∧ is the shear introduced into this member by moment distribution or the sum of the first summations of the end moments for *RO* in Fig. 2–10, i.e., $10 + 9 = 19$. Joints are released for horizontal translation according to the relaxation sequence given on the illustration, Fig. 2–12.

Corrections Resulting in Second Summations. A final summation of the shear *OJ* of Fig. 2–12, for example, shows that the distribution of *H*-forces has changed the shear from +31 to +21 a change of −10. This change must be expressed on the moment distribution sheet, Fig. 2–10, where one finds correction moments of −5 added to each end of this vertical member. Such correction moments are marked with an asterisk. Similarly, on Fig. 2–11 for the *V*-forces we have a shear change in the member *IJ* from −63 to −59 at the first summation. This decrease of +4 is represented by correction moments of +2 (see asterisk) at each end of the member *IJ* in Fig. 2–10.

After these correction moments determined from the shear changes of Figs. 2–11 and 2–12 are introduced on Fig. 2–10, moments are rebalanced and distributed, and the second moment summations are obtained. This process has disturbed the balanced *V*-forces of Fig. 2–11 and also the balanced *H*-forces of Fig. 2–12. It was considered important to recorrect the balance of *V*-forces but not the balance of *H*-forces which are not very significant in this problem Correction shears marked with an asterisk are, therefore, introduced in Fig. 2–11 to make the total shears at that stage in Fig. 2–11 agree with the second summations of moments on Fig. 2–10. These correction shears in Fig. 2–11 are seen to be small. They are quickly balanced to produce equilibrium ($\Sigma V = 0$) at each joint. The final verti-

cal column stresses and vertical shears in the horizontal girders (second summations in Fig. 2–11) are not greatly changed from the first summations.

INTERPRETATION OF RESULTS

Relationship between Grid and Plate Stresses. In Fig. 2–11 we have final values of the direct column stresses. These are plotted as block diagrams for each level in Fig. 2–13. One vertical column corresponds to one-seventh of the cross-sectional area of the plate along a horizontal section. In the horizontal direction there are nine members. Hence, the plate dimensions must bear the relation of 7 to 9. Let us consider a plate $14 \times 18 \times \frac{1}{2}$ in. Each member of the grid then replaces a plate area $2 \times \frac{1}{2} = 1$ sq in. The final direct stresses therefore represent average unit stresses over the $2 \times \frac{1}{2}$-in. plate section. A smooth curve is drawn through the block diagram at the bottom of Fig. 2–13 to show how the unit vertical stress might be expected to vary in the plate.

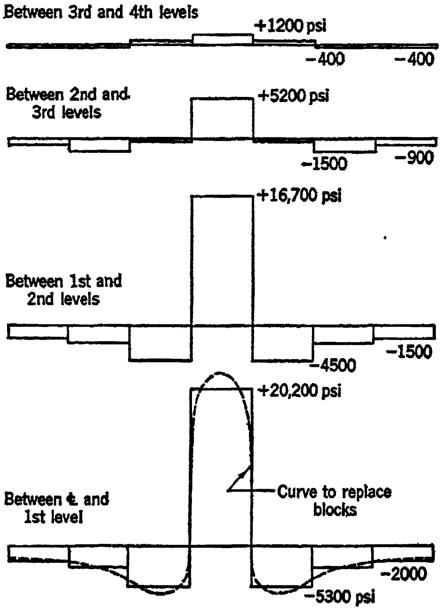


FIG. 2–13. STRESS VARIATION IN THE V-DIRECTION IN GRID AT FOUR LEVELS.

It must be kept clearly in mind that the total stress in any member of this grid represents *an average stress* over an area of plate $2 \times \frac{1}{2}$ in. *The interpretation of these values, therefore, requires the same techniques as the interpretation of strain-gage measurements made on the 2-in. gage lengths of a rectangular grid system.* A plotting of the vertical stress variation in Fig. 2–14 indicates some of the problems of interpretation. It is clear that the questions involved become more easy to answer as the subdivision of the grid is made finer.

Statistical Values of Principal Stresses. If stresses in both horizontal and vertical members are determined with equal care, it is possible to obtain an approximation of the principal stresses for a given point by the following procedure. The shears in the four members meeting at a joint will not be equal but they will approach each other

as the grid is further subdivided. Hence, at any stage in the analysis *the average of these unit shears is the best approximation of the unit shear for a corresponding point in the plate.* By using this average shear with the average of the direct stresses in the adjacent columns and the average of the direct stresses in the adjacent girders, the principal

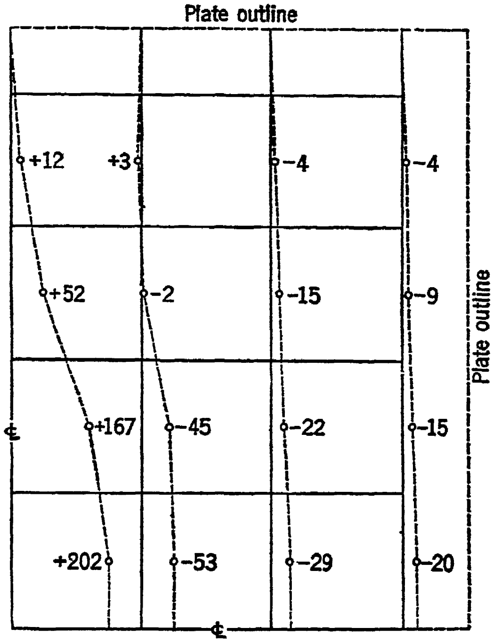


FIG. 2–14. STRESS IN THE V-DIRECTION PLOTTED HORIZONTALLY ON COLUMNS.

stresses may be computed algebraically or graphically. The significance of such stresses will increase as the grid is further subdivided. As was illustrated by Fig. 2–1, it will be recalled that the author does not always place full emphasis upon theoretical point stresses in design. However, a statistical principal stress computed from average grid forces as suggested above appears to be worth study.

Subdivision of the Grid in a Region of Steep Stress Gradients. Although a coarse grid such as the one used in Fig. 2–14 was useful for this particular problem, it may not prove satisfactory for other problems. For example, the study of a bar with a rectangular notch introduces the question of how to handle very steep stress gradients. A solution without long tedious numerical calculations requires subdivision of the grid only in the region of the notch. This problem has been studied and a procedure developed.

In general terms the best procedure is first to analyze the problem by use of a coarse grid. Then consider the grid to be subdivided throughout its entire extent but *use the unit stresses from the coarse grid as a guide for obtaining a first estimate of the direct stresses in the subdivided grid.* Next compute joint movements and the corresponding member shears within the region of steep gradients of stress and balance joint moments in this area. *Follow up with joint force distributions in two directions within the localized region being studied, but assume that the stresses outside of this area remain unchanged from those estimated by use of our knowledge of the stresses in the coarse grid.* Since stress concentrations are always localized, it is possible by this procedure to take advantage of the use of a coarse grid to determine approximate stresses over most of the plate but to gain the advantage of finer subdivision (the equivalent of a short gage length in experimental measurements) within a region of steep gradients of stress.

Conclusion

Usefulness to the Designer. Methods of numerical analysis when applied to plates and wall problems have in the past proved to be very tedious. Such methods might be used in special research studies but they could be of little help to a designer. The physical tool of grid analogy has been simplified by three devices: (1) the concept of permitting initial relative movements of a large group of joints in agreement with statics, called "relative movements," (2) the concept of "strain justification" or checking the accuracy of this initial estimate of stresses and deflections and then revising that estimate as often as necessary to avoid tedious joint relaxations, (3) the procedure of using a finer grid only in regions of steep stress gradients. The first two simplifications are merely refinements of the author's procedures of wind-stress analysis of tall buildings. The third has been used in some form by most writers on numerical methods. These simplifications have reduced the time consumption to the point where a grid analysis for a troublesome problem may ultimately become feasible within the time limitations of a design office. Further simplifications, however, are needed, and some are in prospect.

The concept of a statistical approach to internal stresses is an important one. The fact that grid analogy automatically produces *average stresses over finite widths* rather than point stresses is a favorable factor if used intelligently. It is particularly meaningful that the subdivision of the grid can be related roughly to the stress gradient. Thus it

would be possible to control in a crude way the relationship between the statistical or average stress and the theoretical maximum stress. Finally, grid analogy makes use of *the physical concepts of column stresses, girder moments and shears, column shortening, joint rotation, moment distribution, and joint relaxation* that have become the standard working tools of the structural designer. They are merely put together here in effective form for analysis of the analogous grid.

Appendix A

Grid Deformation under a Diagonal Load. For a grid at 90 degrees to the direction of the loading as shown in Fig. 2–15(*a*) it is evident that each diamond has exactly identical stresses and deformations, as

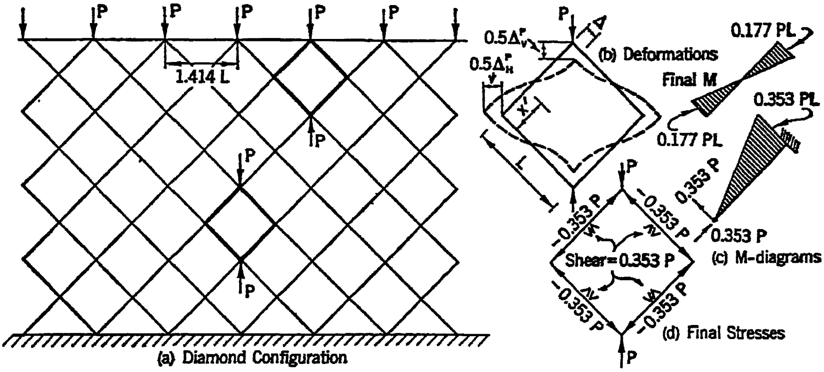


FIG. 2–15. GRID DEFORMATION FOR DIAMOND ORIENTATION.

is shown in Fig. 2–15(*b*). (We are assuming no lateral restraint from the base.) Consider as a static structure a diamond with pins at the ends of its horizontal diameter. Then the indeterminate moment developed at the pin for the condition of free lateral deformation is

$$M_i = \frac{\int M_s\, ds/EI}{\int ds/EI} = \frac{(4/EI)(0.353PL)(L/2)}{4L/EI} = 0.177PL \qquad (2\text{–}14)$$

The deformation along a vertical diameter becomes

$$\begin{aligned} \Delta_V^P &= 4\int_0^L \frac{Mm'\, ds}{EI} + \frac{SuL}{AE} \\ &= \frac{4P}{EI}\int_0^L (0.177L - 0.353x')^2\, dx' + \frac{4(-0.353P)(-0.353)L}{AE} \qquad (2\text{–}15) \end{aligned}$$

$$\Delta_V^P = \frac{0.042PL^3}{EI} + \frac{0.5PL}{AE} \qquad (2\text{–}16)$$

Let $A = Lt$, and $I = \dfrac{1}{12}\dfrac{tL^3}{(1 + \mu)}$ (Equations (2–1) and (2–8))

Then we have

$$\Delta_V^P = \frac{0.5P(1 + \mu)}{Et} + \frac{0.5P}{Et} = \frac{P + 0.5\mu P}{Et} \tag{2–17}$$

Similarly, by reversing the sign of the term $\Sigma SuL/AE$, we obtain the deformation along a horizontal diameter

$$\Delta_H^P = \frac{0.5P(1 + \mu)}{Et} - \frac{0.5P}{Et} = \frac{0.5\mu P}{Et} \tag{2–18}$$

Introducing the Effect of Poisson's Ratio. It will be noticed that Δ_H^P becomes zero when $\mu = 0$, which fulfills one requirement. However, we have not as yet introduced the effect of Poisson's ratio into the grid deformations. The compressive stresses $(-0.353P)$ in one direction in the diamond give rise to tensile elongations at 90 degrees thereto. Evidently these [illegible] of each side of the diamond simply increase its geometrical size. If we again assume that these strains are unresisted, we have

Tensile strain at 90° due to μ for the length L is equal to

$$\mu(0.353P)\frac{L}{AE} = \frac{\mu(0.353P)}{Et}.$$

The corresponding change in either diameter of the diamond is

$$\frac{1.414\mu(0.353P)}{Et} = \frac{0.5\mu P}{Et} \tag{2–19}$$

Combined Strains of Loaded Diamond Including Effect of μ:

$$\Delta_V^P = -\frac{P + 0.5\mu P}{Et} + \frac{0.5\mu P}{Et} = -\frac{P}{Et} = -\frac{PL}{AE} \tag{2–20}$$

$$\Delta_H^P = \frac{0.5\mu P}{Et} + \frac{0.5\mu P}{Et} = \frac{\mu P}{Et} = -\mu(\Delta_V^P) \tag{2–21}$$

Hence, the lateral spread of the diamond is μ times the vertical shortening. Also, the vertical deformation for the diamond arrangement is the same as the vertical deformation for a rectangular grid arrangement. When one notices in Fig. 2–15(a) that the load per unit length is not P/L but $P/1.414L$, it becomes clear that the vertical deformation (PL/AE) which occurs in the depth L of the rectangular grid should,

likewise, occur in a depth of $1.414L$ for the diamond arrangement. Hence, deformations in all directions studied are comparable for the two configurations.

APPENDIX B

NOTCHED BEAM SOLUTION BY GRID ANALOGY

PROBLEM WORKED BY ANDREW J. PYKA *

Correction to Ordinary Beam Theory. In the study of notched beams, the problem of a deflecting base line was encountered and solved. The beam chosen is shown in Fig. 2–16 with the manner of

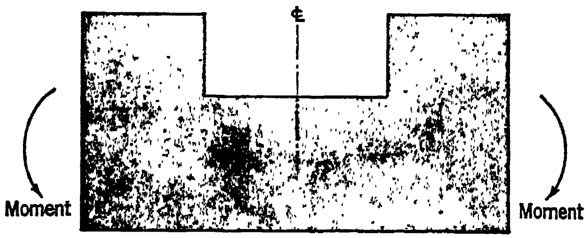


FIG. 2–16. NOTCHED BEAM WITH LOADS.

loading indicated by end couples. This loading when transferred to the analogous grid is applied as joint forces having a straight line variation (following the My/I formula). The corresponding grid for one-half the beam is shown in Fig. 2–17. In order to study the problem as a correction to ordinary beam theory, the boundary forces (Fig. 2–17) are reproduced in the members from the boundaries A and F to the level D where there is an abrupt change of cross section. At the level D, therefore, we have a seriously unbalanced set of joint forces illustrated in Fig. 2–18. Hence, an analysis of grid forces caused by the unbalanced loads of Fig. 2–18 may be added to the ideal stress pattern of Fig. 2–17 to give the final set of grid forces.

Estimates of Joint Movements or of Forces and Shears. The procedure of estimating column and girder forces described earlier in this chapter was followed. Several successive estimates were made and after a balance of moments the estimate was, in each case, improved. For the horizontal or girder forces, these estimates gave rise to horizontal joint movements that had to be computed by summing the shortening or lengthening of members from the deflected base line of

* Engineer, Standard Oil Company of Indiana. This problem is from A. J. Pyka's thesis for the M.S. degree at Illinois Institute of Technology, 1948.

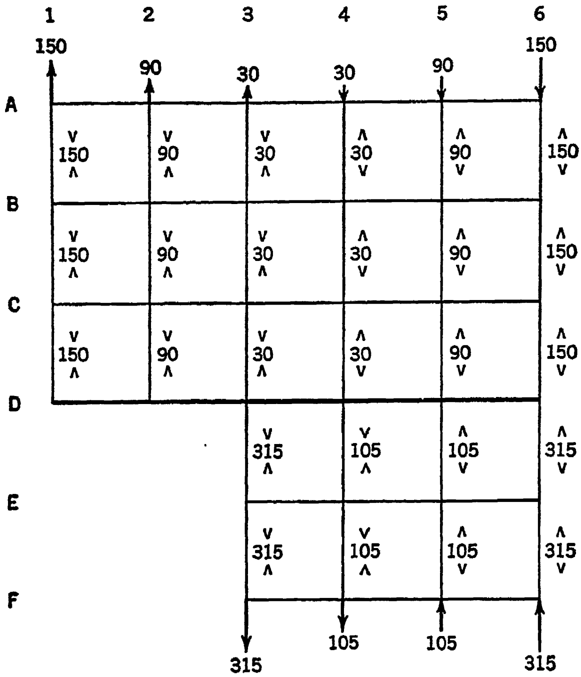


FIG. 2-17. INITIAL FORCES IN MEMBERS.

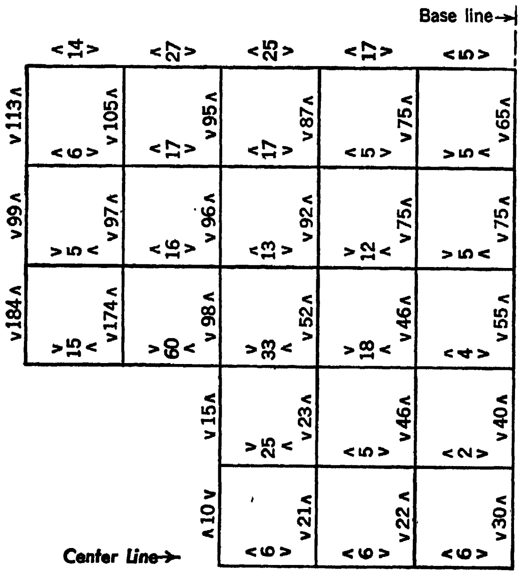


FIG. 2-19. ASSUMED GIRDER FORCES AND CALCULATED FIXED-END SHEARS IN COLUMNS.

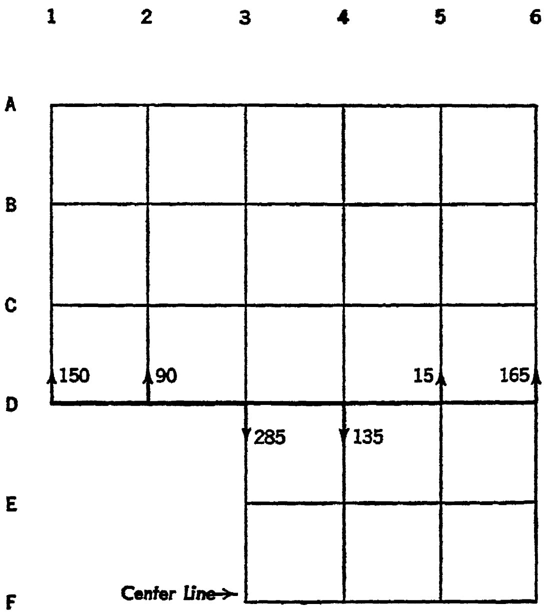


FIG. 2-18. UNBALANCED FORCES AT JOINTS.

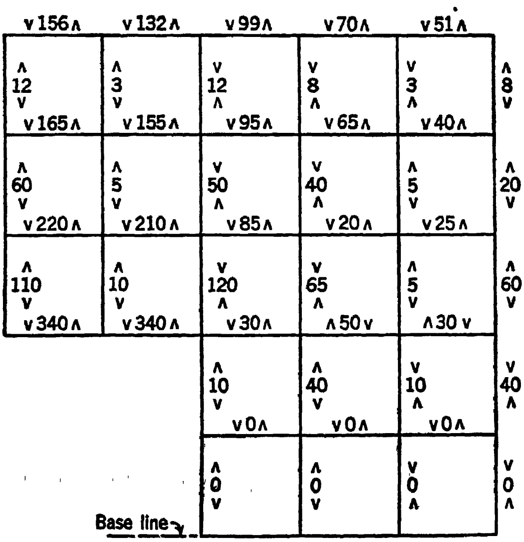


FIG. 2-20. ASSUMED COLUMN FORCES AND CALCULATED FIXED-END SHEARS IN GIRDERS.

Fig. 2–19. A rough estimate of the deflections of the vertical base line was made from the ordinary deflection theory of beams. The deflection estimate was not found to be very satisfactory, however, and *adjustments were made both in the deflections of the base line and in the estimated forces in the girders* to obtain joint equilibrium in the horizontal direction after balancing the joints for moments. The estimated deflections of the vertical base line will be clarified by a study of its fixed-end shears (right-hand column line of Fig. 2–19). Such an estimate may be made in terms of deflections or directly in terms of fixed-end shears as in Fig. 2–19. All other column shears are made dependent thereon.

In Fig. 2–20 the four right-hand columns are attached to the fixed base line (horizontal center line of beam in Fig. 2–20) and the vertical movements of their joints are readily obtained from estimated column forces. However, for the two left-hand columns, estimates of the column forces were made and revised *along with estimates of the shears in the lower horizontal pair of girders* since there is no fixed base line within the depth of the notch from which to work.

Liquidating Unbalanced Joint Forces. In Fig. 2–21 and Fig. 2–22 are shown the smallest sets of unbalanced joint forces obtained through successively estimating the forces in all members, obtaining the joint movements and the corresponding member shears, and then by balancing moments. These unbalanced forces were deemed sufficiently small so that a successive release of joints for vertical and horizontal translation would not prove tedious. After the balance of forces in the horizontal and vertical directions by successive joint translations, the girder forces and column shears of Fig. 2–23, along with the column forces and the girder shears of Fig. 2–24, were obtained. It will be found that these forces and shears are nearly in exact equilibrium at each joint. Accordingly a solution of the grid has been obtained.

Statistical Principal Stresses in the Notched Beam. From the data of Fig. 2–23 and Fig. 2–24, a statistical maximum stress was computed at each joint of the grid. This was accomplished by using the average of the two girder forces as the statistical horizontal stress at the joint and the average of the two column forces as the vertical stress. (Force and stress are interchangeable because each member has a unit area.) The shears of the four members meeting at a joint were averaged to obtain the statistical shear. Then by Mohr's circle or by the comparable mathematical relation, the statistical maximum stress at the joint was obtained.

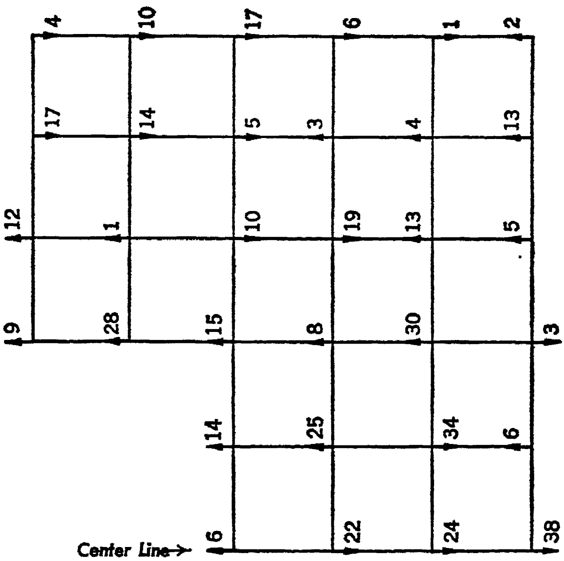


FIG. 2–21. UNBALANCED JOINT FORCES IN THE HORIZONTAL DIRECTION.

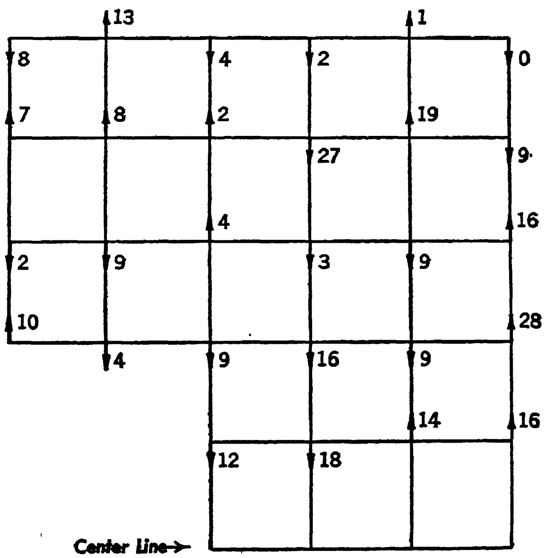


FIG. 2–22. UNBALANCED JOINT FORCES IN THE VERTICAL DIRECTION.

The joint forces shown in Fig. 2–21 and Fig. 2–22 must be liquidated by successive force and moment distributions. Force distribution was performed in two directions resulting in the final stresses and shears of Fig. 2–23 and Fig. 2–24.

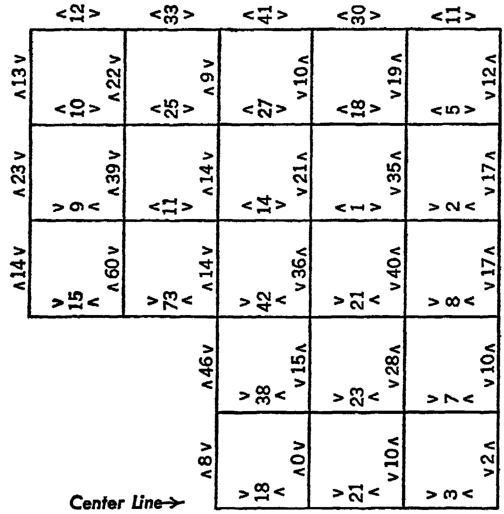


FIG. 2–23. GIRDER FORCES AND COLUMN SHEARS.

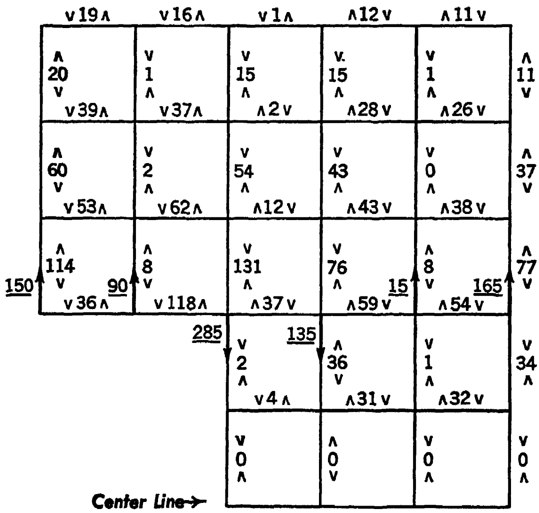


FIG. 2–24. COLUMN FORCES AND GIRDER SHEARS.

Fig. 2–23 and Fig. 2–24 represent the final direct stresses and shears in vertical columns and horizontal beams. Each joint will be found to be nearly in exact equilibrium when checked by the equations of statics. The remaining joint restraints are therefore neglible. The stresses and shears of Figs. 2–23 and 2–24 give rise to the statistical principal stresses plotted in Fig. 2–25.

Stress Contours. These values for all joints are recorded on Fig. 2–25 and approximate stress contours are drawn to illustrate the stress pattern in the beam. It is recognized that a finer division of the grid would be necessary to plot such curves with the desired accuracy.

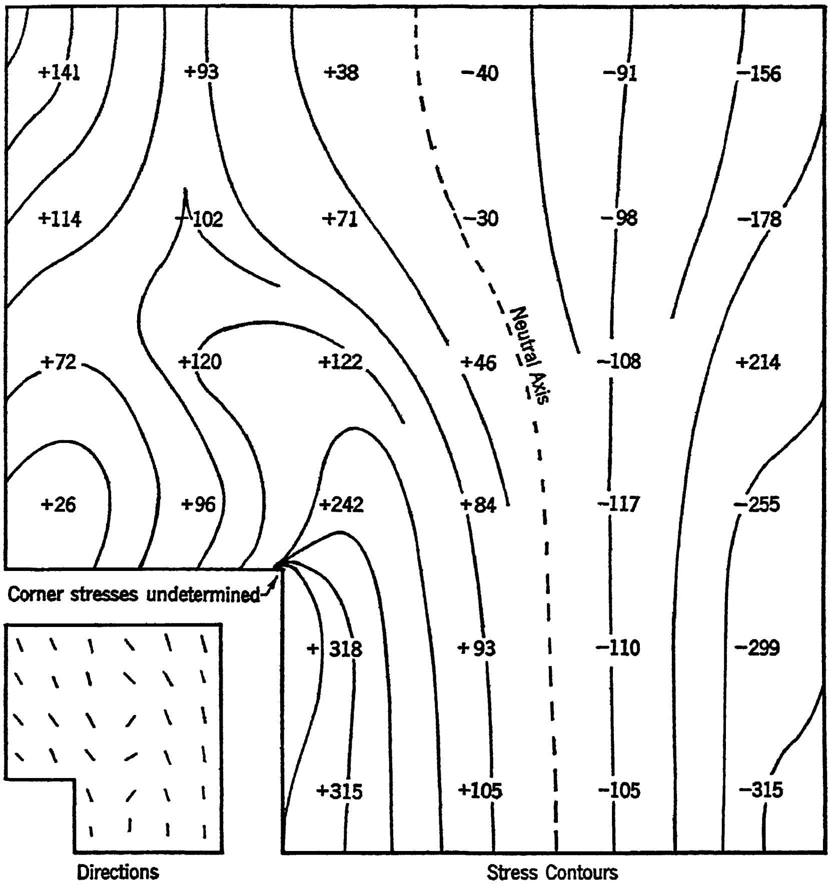


FIG. 2–25. MAXIMUM PRINCIPAL STRESSES DETERMINED STATISTICALLY.

A photoelastic pattern obtained by Dr. A. J. Durelli and analyzed by S. J. Fraenkel and R. L. Janes gave *p–q* values (shears) which checked the statistical shears from this grid analysis within about 20 per cent. Considering the crude subdivision of the grid, the agreement is rather good. No comparison was possible near the corner of the notch.

Extrapolation to the boundary was considered reasonable, but the curves of constant stress are seen to be free from the critical internal corner where a finer subdivision of the grid will be needed because of the steep stress gradients. After this analysis had been completed a photoelastic study of this beam was made. A comparison of *p–q* stresses (shears) showed a variation of about 20 per cent from the

statistical shears. Considering the crudity of the grid, this agreement seems rather good. Of course, no useful comparison could be made close to the corner of the notch.

References to Supporting Publications of L. E. Grinter

(1) Discussion of Hardy Cross' paper on moment distribution, Trans. ASCE **96,** 11–20 (1932).
(2) "Wind stress analysis simplified," Trans. ASCE (1934), pp. 610–634.
(3) "Sign conventions for moment distribution," Eng. News-Record (Mar. 7, 1935), p. 359.
(4) "Analysis of continuous frames by balancing angle changes," Trans. ASCE **102,** 1020–1036 (1937).
(5) "Determining influence lines by balancing angle changes," Eng. News-Record (Oct. 6, 1938), pp. 443–445.
(6) "Direct design of continuous beams," Eng. News-Record (Mar. 16, 1939), pp. 50–52.
(7) *Theory of Modern Steel Structures* (Macmillan, 1937 and 1949), Vol. 2. See chapter on "Modern Methods of Analysis of Continuous Frames" and chapter on "Analysis of Continuous Frames by Balancing Angle Changes."
(8) *Automatic Design of Continuous Frames in Steel and Reinforced Concrete* (Macmillan, 1939). Numerical methods are here applied both to analysis and to design.

CHAPTER 3

NUMERICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS BY RELAXATION METHODS *

F. S. SHAW **

Synopsis. Many of the present day problems with which the engineer or physicist is confronted lead, in one way or another, to a two-dimensional linear partial differential equation of the boundary value type. For a few simple mathematical shapes, e.g., circles, squares, ellipses, etc., exact solutions are available; however, in general these are not profiles of great interest.

In an endeavour to provide means for obtaining approximate solutions of the problems of practical importance, the numerical methods of solution known as relaxation methods have been developed in recent years by R. V. Southwell and others. *In effect the partial differential equation concerned is replaced by its finite difference equivalent, so that instead of having to solve one governing equation the problem reduces to that of solving a large number of simultaneous algebraic equations.* The equations are always simple, though the number may be of the order of one or two hundred.

This paper contains nothing that is new, but attempts to make clear the use of many of the devices employed in solving such problems, for example, lines of symmetry, block operators, curved boundary operators, etc. It will be seen that the methods, as well as being extremely simple, are also capable of very general application.

INTRODUCTION

Associated with many of the problems that arise in engineering or physics is the necessity of solving a linear partial differential equation of the boundary value type.

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Typical Equations. Consider, for instance, Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \text{ i.e., } \nabla^2\phi = 0 \tag{3-1}$$

or Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x,y), \text{ i.e., } \nabla^2\phi = f(x,y) \tag{3-2}$$

where the function $f(x,y)$ is given. These two equations * are associated with many fields of study: elasticity, electricity and magnetism, hydrodynamics, etc. *The function ϕ, or its normal derivative, $\partial\phi/\partial n$, is specified around the boundary of some well-defined domain, and it is necessary to solve for ϕ inside the domain, such that either equation (3–1) or (3–2) is satisfied there.*

In cylindrical coordinates equations of the type

$$\frac{\partial^2\psi}{\partial z^2} + \frac{K}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = 0 \tag{3-3}$$

are met with, of which the one associated with torsion of a shaft of rotational symmetry:

$$\frac{\partial^2\psi}{\partial z^2} - \frac{3}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = 0 \tag{3-4}$$

is a well-known example.

Fourth Order Equations. Another common type of equation is that involving the biharmonic operator. In the field of elasticity there occurs both

$$\frac{\partial^4\chi}{\partial x^4} + 2\frac{\partial^4\chi}{\partial x^2\,\partial y^2} + \frac{\partial^4\chi}{\partial y^4} = 0 \tag{3-5}$$

and

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = f(x,y) \tag{3-6}$$

These equations, being of the fourth order, have boundary conditions more complicated than those associated with equations (3–1) to (3–4). Those most commonly experienced have both χ and $\partial\chi/\partial n$ specified, or w and $\partial^2 w/\partial n^2$, or other similar combinations. Equation (3–5) is associated with the problems of plane stress or plane strain, and equation (3–6) with the deflection of a thin flat plate loaded normal to the plane of the plate.

* Problems in two independent variables are the only ones that will be considered.

All of the above equations pertain to isotropic media; however, similar but more complicated equations may be written down for anisotropic media. All the equations are linear.

Solutions Sought. For various simple mathematical shapes, e.g., circles, ellipses, squares, these equations have been solved exactly, and complete solutions are available. This situation is not sufficient, however, for the engineer or physicist. He is often concerned with shapes that are not expressible in simple mathematical form, and for his problems it does not seem possible to obtain solutions to the above equations by orthodox means. Be that as it may, an answer is still desired, and an answer giving stresses, say, to within 5 per cent or even 10 per cent is much better than nothing.

In an endeavour to fulfill this general need, the approximate numerical methods known as relaxation methods have been and are still being developed, in England mainly by R. V. Southwell [1] and an associated group, and it is with a few of the general techniques associated with the methods of Southwell that this paper is concerned.

Without discussing any particular physical problem it is hoped to show how the methods work.* It is essential to realize that the whole treatment is numerical, that is, any particular solution consists of a pattern of numbers, so that as such each separate problem must be solved anew. No formula or general expression becomes available into which particular dimensions can be inserted to give answers to particular problems.

Finite Difference Equations

Relationship to Differential Equations. Two major steps are used in obtaining a numerical solution. Firstly the governing partial differential equation is replaced by its finite difference approximation. If, then, a regular mesh of lines is superimposed on the domain of the problem being considered, *it is possible to write down one finite difference equation for each node of the network.* Thus, instead of being required to solve one partial differential equation, the problem reduces to solving a set of simultaneous finite difference equations, the latter being regarded as the second step.

Since the number of equations involved will be large, often of the order of 100 to 200, solution by successive elimination, or by the use

* The material from which this paper is drawn has already appeared in the form of a more comprehensive report, see reference 2.

of determinants or matrix methods, is not feasible. It is to the various devices used in actually obtaining a solution that the name *relaxation methods* has been applied.

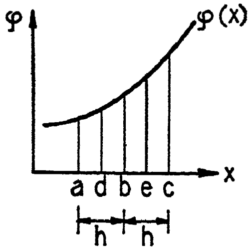

FIG. 3-1. FUNCTION OF $\phi(x)$.

Setting up Difference Equations. In Fig. 3-1, let the curve shown be the graph of some function $\phi = \phi(x)$. At some point $x = x_e$, we have by definition for the point e of Fig. 3-1

$$\left(\frac{d\phi}{dx}\right)_e = \lim_{h\to 0} \frac{\phi\left(x_e + \frac{h}{2}\right) - \phi\left(x_e - \frac{h}{2}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\phi_c - \phi_b}{h}$$

where by $\left(\frac{d\phi}{dx}\right)_e$ is meant the value of $\frac{d\phi}{dx}$ at the point $x = e$, etc. Consequently, when h is small, we may write for equation (3-1)

$$\left(\frac{d\phi}{dx}\right)_e \approx \frac{\phi_c - \phi_b}{h} \tag{3-7}$$

and similarly

$$\left(\frac{d\phi}{dx}\right)_d \approx \frac{\phi_b - \phi_a}{h} \tag{3-8}$$

Also, since

$$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(\frac{d\phi}{dx}\right)$$

it follows that

$$\left(\frac{d^2\phi}{dx^2}\right)_b \approx \frac{\left(\frac{d\phi}{dx}\right)_e - \left(\frac{d\phi}{dx}\right)_d}{h}$$

By introducing values from equations (3-7) and (3-8) one obtains

$$\left(\frac{d^2\phi}{dx^2}\right)_b \approx \frac{\phi_c + \phi_a - 2\phi_b}{h^2} \tag{3-9}$$

In equation (3-9) the expression $(\phi_c + \phi_a - 2\phi_b)/h^2$ is the finite difference approximation for $(d^2\phi/dx^2)_b$.

Check by Taylor's Series. Alternatively, on the assumption that locally the function ϕ may be expanded in the form of a Taylor's series, we may write

$$(\phi)_x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \tag{3-10}$$

With $x = x_b$ as origin we then obtain

$$a_0 = \phi_b$$
$$a_1 = \left(\frac{d\phi}{dx}\right)_b$$
$$2!\,a_2 = \left(\frac{d^2\phi}{dx^2}\right)_b$$
$$3!\,a_3 = \left(\frac{d^3\phi}{dx^3}\right)_b$$
etc.

and, substituting $(x_c - x_b) = h$, $(x_a - x_b) = -h$, etc., and solving for a_2 we obtain

$$2!\,a_2 = \left(\frac{d^2\phi}{dx^2}\right)_b = \frac{\phi_c + \phi_a - 2\phi_b}{h^2} + \frac{h^2}{12}\left(\frac{d^4\phi}{dx^4}\right)_b + \cdots \tag{3-11}$$

so that the finite difference approximation for $d^2\phi/dx^2$ as given by equation (3–9) neglects terms $(h^2/12)(d^4\phi/dx^4) + \cdots$

Difference Equivalents. Consider now the function $\phi = \phi(x,y)$ as a function of two independent variables. If, on the x,y plane, a mesh of lines be drawn parallel to the x and y axes, and spaced a distance h apart, as in Fig. 3–2, then in the manner just outlined the following approximations are obtained:

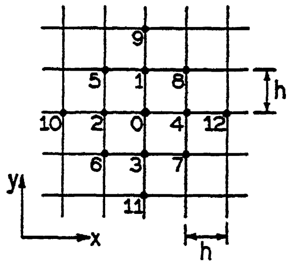

FIG. 3–2. STANDARD LATTICE.

When points 2 and 4 are $2h$ apart

$$\left.\begin{aligned}\left(\frac{\partial\phi}{\partial x}\right)_0 &\approx \frac{(\phi_4 - \phi_2)}{2h}\\ \left(\frac{\partial\phi}{\partial y}\right)_0 &\approx \frac{(\phi_1 - \phi_3)}{2h}\end{aligned}\right\} \tag{3-12}$$

From equation (3–9)

$$\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)_0 \approx \frac{(\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0)}{h^2} \tag{3-13}$$

Similarly

$$\begin{aligned}&\left(\frac{\partial^2\psi}{\partial x^2} - \frac{3}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}\right)_0\\ &\quad\approx \frac{(\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0)}{h^2} - \frac{3(\psi_1 - \psi_3)}{2r_0h}\\ &\quad= \frac{1}{h^2}\left\{\psi_1\left(1 - \frac{3h}{2r_0}\right) + \psi_2 + \psi_3\left(1 + \frac{3h}{2r_0}\right) + \psi_4 - 4\psi_0\right\}\end{aligned} \tag{3-14}$$

One may also write

$$\left(\frac{\partial^4\chi}{\partial x^4} + 2\frac{\partial^4\chi}{\partial x^2\,\partial y^2} + \frac{\partial^4\chi}{\partial y^4}\right)_0$$
$$\approx \frac{1}{h^4}\{20\chi_0 - 8(\chi_1 + \chi_2 + \chi_3 + \chi_4) + 2(\chi_5 + \chi_6 + \chi_7 + \chi_8) + (\chi_9 + \chi_{10} + \chi_{11} + \chi_{12})\} \quad (3\text{–}15)$$

The notation follows that of equations (3–1), (3–4) and (3–5).

Residual and Relaxation Operators

For clarity it is advisable to discuss a particular equation; accordingly the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x,y) \quad (3\text{–}16)$$

will be considered, the function $f(x,y)$ being given.

Use of Residuals. Let Fig. 3–3 represent a portion of the area involved, the points being numbered for convenience of discussion only. Consider point 13. Using the approximations already developed the finite difference equation appropriate to that point as controlled by equation (3–16) is

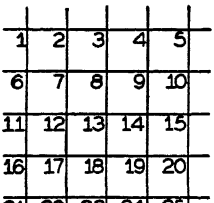

Fig. 3–3. Grid for Equation (3–16).

$$\phi_8 + \phi_{12} + \phi_{18} + \phi_{14} - 4\phi_{13} = h^2 f(x,y)_{13} \quad (3\text{–}17)$$

There will be one such equation for each internal mesh point.

Suppose now that by a guess (or by some other means) values have been obtained for all of the ϕ_i, $i = 1, 2, \cdots$. In general they will be incorrect. Accordingly, the equations of type (3–17) must be written as follows

$$\phi_8 + \phi_{12} + \phi_{18} + \phi_{14} - 4\phi_{13} - h^2 f(x,y)_{13} = R_{13} \quad (3\text{–}18)$$
$$\text{etc.}$$

where the R's are measures of the errors. This particular equation is for points surrounding point 13. For the correct values of ϕ, the R's, or residuals as they are called, will be zero everywhere. For values of ϕ which are not in serious error the residuals will *all* be small, whilst for values of ϕ that are far from correct some or all of the residuals will be large. Thus, for instance, if ϕ_i is taken as zero everywhere, $R_i = h^2 f(x,y)_i$. Our aim then is to reduce each R to a value which will be zero or very close to it.

Relaxation Operators. Consider again equation (3–18). If ϕ_8 is altered by $+1$, then R_{13} is altered by $+1$, and similarly for unit alterations to ϕ_{12}, ϕ_{18} or ϕ_{14}. If however ϕ_{13} is altered by $+1$, then R_{13} is altered by -4. Note that this type of result is true for each of the equations like (3–18) centered at each of the various internal mesh points. We thus have a relaxation operator, that is, an operator by means of which the residual at any point can be systematically modified. If the value of ϕ at any internal point be altered by $+1$, the residual at that point is altered by -4; at the same time the residuals at each of the four surrounding points are altered by $+1$. Such a relaxation operator is independent of the particular point being considered.

Two Types of Operators. Diagramatically the two types of operators are as shown in Figures 3–4 and 3–5. From a consideration of the original or estimated ϕ's at the five related points (Fig. 3–4) the

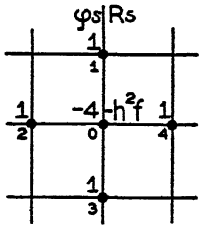

FIG. 3–4. RESIDUAL OPERATOR.
($R_0 = 1\phi_1 + 1\phi_2 + 1\phi_3 + 1\phi_4 - 4\phi_0 - h^2f$)

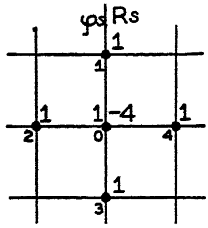

FIG. 3–5. RELAXATION OPERATOR.
(Result on Residuals of $\Delta\phi_0 = +1$)

residual at the central point is obtained. This *residual operator* (Fig. 3–4) gives us the first residual at 0, i.e. R_0 as $1\phi_1 + 1\phi_2 + 1\phi_3 + 1\phi_4 - 4\phi_0 - h^2f(x,y)_0$ which reduces to $-h^2f$ when the initial ϕ's are taken to be zero. By altering the ϕ at the central point 0 in Fig. 3–5 (and each internal point is a central point for some particular group of five points) the residual at that point is modified, as also to a lesser extent (25 per cent) are the residuals at the four surrounding points. This operator, Fig. 3–5, is the *relaxation operator.* Figures 3–4 and 3–5 therefore give the appropriate ϕ multipliers and the resulting R effects. Obviously, in using the operators, superposition holds.

THE RELAXATION PROCESS

Example, Fig. 3–6. By way of an example consider the problem of solving the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x,y) \qquad (3\text{–}16)$$

for the square domain $0 \leq x \leq 4$; $0 \leq y \leq 4$. The function $f(x,y)$ is given, and shall here be taken as constant, say -100. On the boundary let ϕ be given as zero. The problem is as shown in Fig. 3–6.

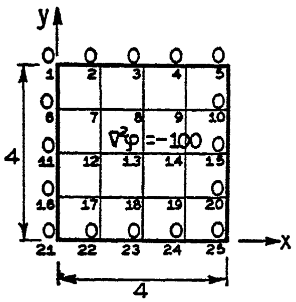

FIG. 3–6. MESH STUDY OF EQUATION (3–16).

Let the square domain be subdivided into squares of side length $h = 1$, and let each node of the network be numbered as shown in the figure, numbered nodes being useful only for the purpose of discussion. The mesh is deliberately made coarse here, and in practice would be spaced much finer. In carrying out the relaxation process it is convenient to draw the actual problem to scale on tracing paper which will withstand repeated erasure. The scale should be such that the spacing of the mesh is at least one inch. Increments of ϕ are written on the left of the vertical mesh lines, and to the right are recorded the progressive totals of the residuals.

First Estimate of ϕ's. For want of something better the initial guessed solution will be taken as $\phi = 0$ everywhere, so that each initial residual is of magnitude 100. Several consecutive steps in the relaxation process are shown as steps 1 to 8 of Fig. 3–7. *At each step, using the relaxation operator, the largest residual is reduced approximately to zero, symmetry being preserved.* Since the boundary values of ϕ are not to be altered, the residuals at the boundary points are not recorded. In each step * an asterisk has been placed before the added increment of ϕ and behind each new residual.

Convergence. Several points arise out of this problem that warrant consideration. In step (1) of Fig. 3–7 the total of all the residuals is 900; in step (2), although the central residual has been completely liquidated, the total is still 900. This is a natural feature of this type of problem. From the relaxation operator it is obvious that the net change in the residuals, for a "displacement" of any point other than one adjacent to the boundary, is zero. *The total of the residuals is, in fact, only reduced by pushing some of them "over the boundary." Any other operation merely has the effect of "smearing" the residuals over a larger area than that originally occupied.* This is obvious from step (3). Our aim then is always to move the residuals towards the boundary

* As the columns of figures grow it becomes necessary to erase them, and replace them by corrected figures; on the left of any vertical line this will be the total of all the increments, on the right it will be merely the last number written down.

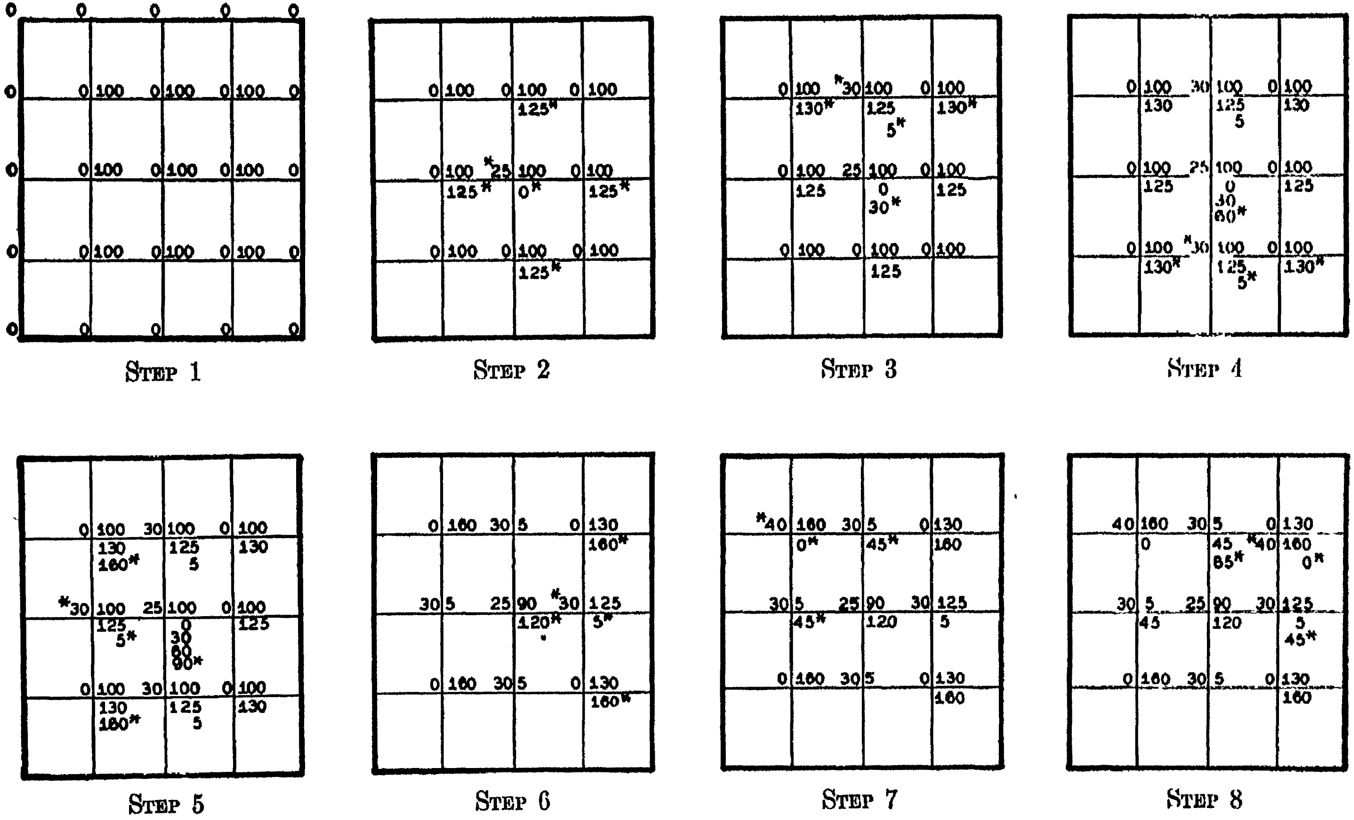

Fig. 3-7. Steps in the Relaxation Procedure.

* An asterisk is placed before each increment of ϕ and behind each new residual.

so that, finally, displacements of points adjacent to the boundary will reduce their total. This is one factor limiting the speed of convergence.

It will also have been noticed that the residuals have a tendency to "wash back again"; compare, for instance, step (2) with step (6) which shows that the corner residuals have increased. This indicates that it would be advantageous to over-liquidate the residuals, actually changing their sign. Residuals washing back would then tend to be canceled out. The question how much to over-relax can only be answered by experience.

Repeated Operations. One other point suggests itself. In steps (3), (4), (5) and (6), to a certain extent the same relaxation operation is being repeated four times. Corresponding points are being displaced by the same amount. *Considerable time would be saved by the use of an operator by means of which all four points could be displaced simultaneously.* This introduces the concepts of line and block operators, and also the use of any lines of symmetry that might be inherent in the problem.

Line and Block Relaxation Operators

Consider the effect of the simultaneous displacement, each by the same amount, of two adjacent points. Obviously in the first instance this may be obtained by writing down separately the effects of each displacement and then adding them together. Continuing the use of unit operators, the two-point relaxation operator is thus obtained, Fig. 3–8.

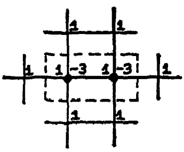

Fig. 3–8

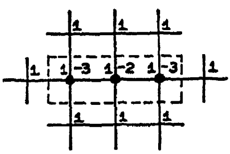

Fig. 3–9

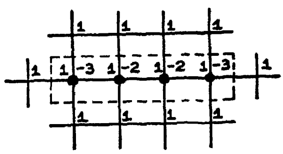

Fig. 3–10

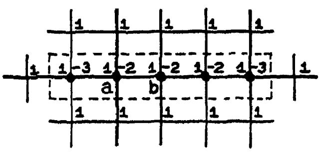

Fig. 3–11

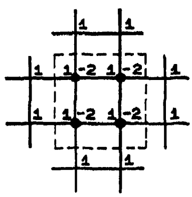

Fig. 3–12

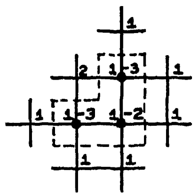

Fig. 3–13

Fig. 3–8 to 3–11; Line Operators. Fig. 3–12 and 3–13; Block Operators.

In the same manner are obtained the three, four and five point unit line operators given in Figures 3–9, 3–10 and 3–11. From them it is

possible to derive a working principle whereby all such operators can be immediately written down by inspection: Follow Fig. 3–11.

Rule for Writing Operators. *All points like* a, *having an alteration to the residual of −3, are connected directly to three points that remain undisplaced, whilst all points like* b, *having a residual alteration of −2, are connected directly to two points that remain undisplaced. Of the points that do not move, at any one such point there is an alteration to the residual of +1 for each* **direct** *connection from that point to a point that is being displaced.*

Examples of Block Operators. Equally simply, it is possible to construct relaxation operators in which groups of points are displaced simultaneously and by the same amount. Figures 3–12 and 3–13 serve as examples.

As a final example Fig. 3–14 gives a large irregular unit block operator. Its correctness may be checked both by adding the separate effects of each point, and by "counting strings." Inspection of Fig. 3–14 shows the advantage of such types of relaxation blocks; for whilst points on the boundary of the block do experience alterations in the residuals, *points inside the block suffer no change whatever.* To a large extent then, judicious use of such blocks can prevent much of the "washing back" of residuals. As before, the total of all the residual changes caused by such blocks is zero. Their use is simply to push residuals out to the boundary as rapidly as possible and aid convergence.

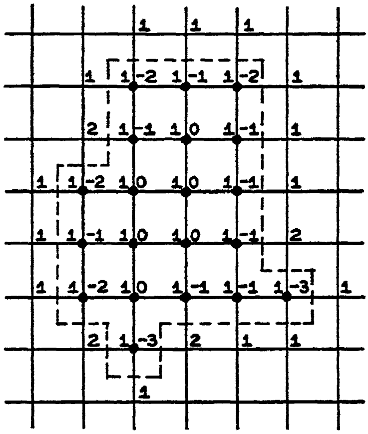

FIG. 3–14. IRREGULAR BLOCK OPERATOR.

Consideration of the total residual change inside a block, and of the residual total of the points to be relaxed by means of the block, enables the magnitude of the block multiplier to be calculated. High accuracy in the multiplier is unnecessary.

Lines of Symmetry. In many problems the regions concerned will exhibit one or more lines of symmetry, so that in solving for the unknowns over the complete area there is unnecessary duplication of effort. Thus, since in the previous example of the square there is eight-fold symmetry, it is sufficient to find a solution inside the area

bounded by either a vertical or horizontal centerline, a diagonal, and the enclosing portion of the boundary (Fig. 3–22), with the remainder of the solution following by repetition.

Techniques with Symmetry. No markedly new technique is involved in solving this latter type of problem. It is necessary merely to remember to *preserve symmetry of displacement alteration each time a point adjacent to a line of symmetry is displaced.* That is, such an operation is accompanied by an automatically equal displacement of the corresponding point on the other side of the line of symmetry. As a result, a point on the line of symmetry will receive a change in its residual from both displacements.

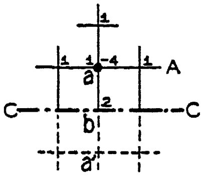

FIG. 3–15. ADJUSTING RELAXATION OPERATOR FOR SYMMETRY.

Figure 3–15 will make this clear. Let C–C be a line of symmetry, and let the region A be that portion of the problem which is being solved. Suppose point a is displaced by $+1$; as a consequence so also is point a', although its displacement is not being recorded of course. Hence point b receives a residual alteration of $+2$, so that the relaxation operator for points like a is that given in the figure.

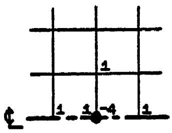

FIG. 3–16

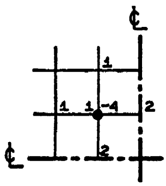

FIG. 3–17

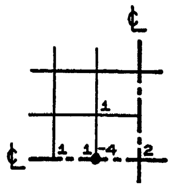

FIG. 3–18

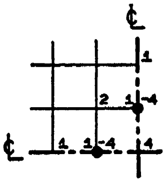

FIG. 3–19

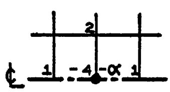

FIG. 3–20

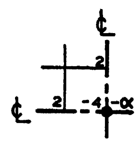

FIG. 3–21

FIG. 3–16 TO FIG. 3–19. SYMMETRICAL RELAXATION OPERATORS.
FIG. 3–20 AND FIG. 3–21. SYMMETRICAL RESIDUAL OPERATORS FOR FIRST VALUES OF R.

In similar manner Figures 3–16, 3–17, 3–18 and 3–19 are examples of other such operators.

Similar considerations also hold for the residual operators. With $\alpha = h^2 f(x,y)$, where $f(x,y)$ is the given function in equation (3–2) [i.e. $\nabla^2\phi = f(x,y)$], the following operators are obtained, Fig. 3–20 and Fig. 3–21.

The Problem of the Square Reworked

Using the foregoing ideas the problem previously introduced now becomes trivial although in practice problems are rarely so simple as this. The solution which makes use of the steps of Fig. 3–22 follows:

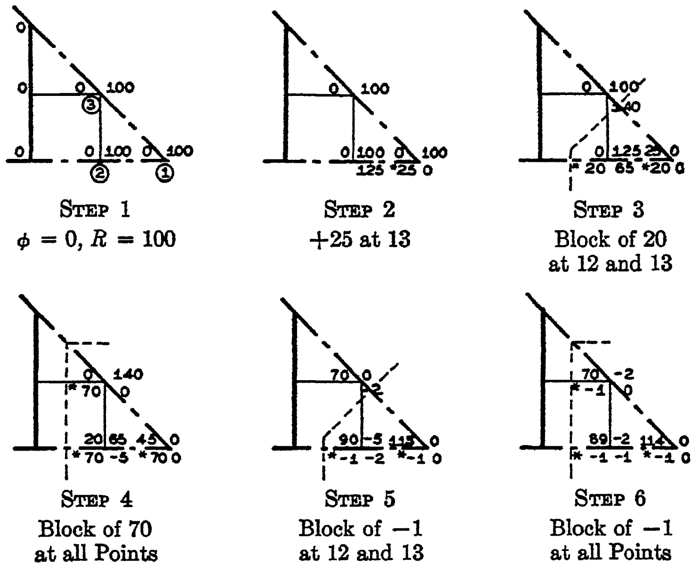

Fig. 3–22. Six Steps in the Use of Block Operators.

Note that the square has been renumbered in Fig. 3–22. Point 1 is the point 13 of Fig. 3–6, point 2 corresponds to 12, and point 3 corresponds to point 7 of Fig. 3–6.

Step (1) Starting solution is $\phi = 0$ everywhere, so that the residuals are all 100.

Step (2) A displacement of 25 on the central point 1 (which is 13 of the square of Fig. 3–6) gives a resultant residual there of zero.

Step (3) Put a block of 20 on points 1 and 2. Note that a complete block on all points, i.e., 1, 2, and 3, would have tended to over-relax point 3 and under-relax point 2. It is better therefore to relax only partially point 2 and so obtain a better distribution of residuals at points 2 and 3. These may then be handled by another block.

Step (4) A block of 70 on all points.

Step (5) A block of -1 on points 1 and 2. The reason is essentially the same as that in *Step 3*.

Step (6) A block of -1 on all points.

The complete solution then follows as in Fig. 3–23.

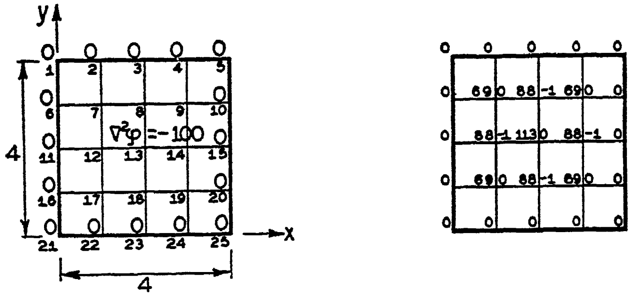

FIG. 3–23. FINAL NUMERICAL SOLUTION — PROBLEM OF FIG. 3–6.

CURVED BOUNDARIES

In many of the boundary value problems encountered in practice the boundaries are curved, so that, adjacent to the boundary the foregoing treatment is incorrect. Let Fig. 3–24 represent one such problem.

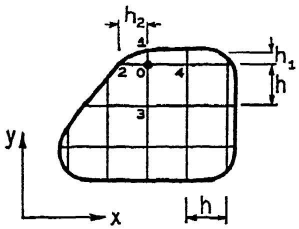

FIG. 3–24. CURVED BOUNDARY PROBLEM.

Correcting the Operator. On covering the region with a square mesh, points like that marked 0 occur, that is, points in which one or more of the associated *arms* 01, 02, 03, or 04, is less than the standard length h. For such points the standard unit operators are incorrect, and it becomes necessary to develop special operators.

Consider the polynomial

$$\phi = \phi_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy \tag{3–19}$$

Thus, at the point 0 ($x = 0$, $y = 0$), we have

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 = 2a_3, \quad \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 = 2a_4 \tag{3–20}$$

Then, referring to Fig. 3–24, substituting the irregular arms h_1 and h_2 and solving for a_3 and a_4 we find that

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 = 2\,\frac{h(\phi_1 - \phi_0) + h_1(\phi_3 - \phi_0)}{hh_1(h + h_1)} \tag{3–21}$$

and

$$\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 = 2\,\frac{h(\phi_2 - \phi_0) + h_2(\phi_4 - \phi_0)}{hh_2(h + h_2)} \tag{3–22}$$

so that $(\nabla^2\phi)_0$ may be written as

$$h^2(\nabla^2\phi)_0 = \frac{2}{\alpha_1(1+\alpha_1)}\phi_1 + \frac{2}{(1+\alpha_1)}\phi_3 + \frac{2}{\alpha_2(1+\alpha_2)}\phi_2 + \frac{2}{(1+\alpha_2)}\phi_4 - \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2}\right)\phi_0 \tag{3-23}$$

where $\alpha_1 = h_1/h$, $\alpha_2 = h_2/h$. Hence for $0 < \alpha < 1$ we have

$$h^2(\nabla^2\phi)_0 = A\phi_1 + B\phi_2 + C\phi_3 + D\phi_4 - (E+F)\phi_0 \tag{3-24}$$

where the values of $A, \cdots, F$ may be calculated and tabulated * once and for all.

Revised Residual Operator. This finite difference equivalent for the harmonic operator may then be used in precisely the same manner as the standard (−4, 1, 1, 1, 1) operator. Equations like (3–17) become

$$A\phi_1 + B\phi_2 + C\phi_3 + D\phi_4 - (E+F)\phi_0 - h^2f(0) = R_0$$

so that, diagrammatically, for an irregular star centered at point a_0, the residual operator is as in Fig. 3–25, where the numbers $A_a, \cdots, F_a$, will all be known.

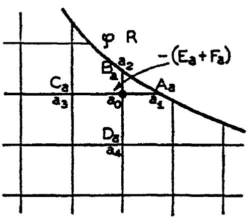

FIG. 3–25

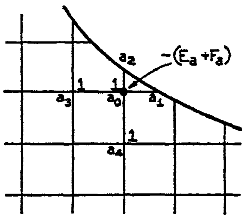

FIG. 3–26

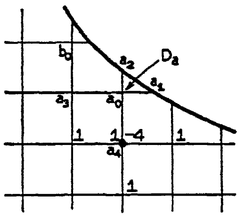

FIG. 3–27

FIG. 3–25. RESIDUAL OPERATOR.
FIG. 3–26 AND FIG. 3–27. RELAXATION OPERATORS NEAR A CURVED BOUNDARY.

Revised Relaxation Operators. The relaxation operator is, however, a little more tricky. Thus, if a unit displacement is made to the ϕ at point a_0, i.e., to ϕ_0, the residual there is altered by $-(E_a + F_a)$. At the same time the stars centered at points a_3 and a_4 (Fig. 3–26), being ordinary stars, each receive an alteration to their residual of +1. No residuals are recorded at the boundary points a_1 and a_2. The relaxation operator for point a_0 is, accordingly, as given in Fig. 3–26.

In like manner, the relaxation operator may be constructed for the star centered at point a_4. Since a unit alteration made to the value of ϕ at point a_4, i.e., ϕ_4, will alter the residual at point a_0 by amount

* These are available in reference 3, p. 65.

D_a the operator is as in Fig. 3–27. Note however that to obtain the relaxation operator for point a_3 of the a_0 star (which is also point b_4 for the star centered at b_0) requires the evaluation of the residual operator for the star centered at point b_0. This follows since a unit alteration to the value of ϕ at point a_3 will alter the residual at point b_0 by amount D_b. At the same time, of course, the residual at point a_0 is altered by amount C_a. Given the b_0 residual operator also, then the relaxation operator for point a_3 (or b_4) may be written down in the same manner as for point a_4.

With all these irregular stars evaluated for a particular problem, the relaxation process may then be carried out in the manner already described. The presence of the irregular stars adds a little to the labor but nothing to the complexity of the method, and a little practice makes their use automatic.

Other Auxiliary Devices. The foregoing covers the main features of the relaxation process; however other devices that are used include the following:

Nondimensional Equations. In virtue of the numerical answers obtained, it is possible to solve only one particular problem at a time. Some generalization is possible, however (reference 4). "The governing equations, as so far treated, are dimensional in that x and y are coordinates measured in some particular units. By choosing some representative dimension, L say, and using the dimensional relations

$$x = \zeta L, \quad y = \eta L, \cdots$$

the differential equation may be made nondimensional. The solution then found will apply to all regions of a particular shape and associated with a similar function $f(x,y)$ [e.g., from $\nabla^2\phi = f(x,y)$], without restriction on the size, or on the absolute magnitude of the function."

Finer Nets. It is possible to use different mesh sizes in different parts of the same problem. In this way attention may be concentrated on certain local areas of the problem. For more information on the above points see the papers mentioned in the reference list. Also see Chapter 4.

Remarks on Other Boundary Value Problems. Whilst the foregoing discussion has dealt fairly exhaustively with the harmonic operator, for other types of problems special devices have been developed. Thus, for eigenvalue problems "optimal synthesis" is very helpful. In some of the problems involving the biharmonic operator a "45 line" technique assists convergence, and for the same class of problems

additional difficulties arise in the satisfaction of the boundary conditions. These points cannot be discussed here; however, reasonably complete details are available in the relevant papers.

Poisson's Equation. It is obvious that the general techniques already discussed will carry over with little if any modification to other classes of problems. By a simple alteration all of our remarks apply directly to Poisson's equation (3–2), for it merely requires the deletion of the term $h^2f(x,y)$ from the residual operator.

Varying Operators. For problems of the type covered by equation (3–3), since the finite difference equation contains a term involving $1/r$, it is obvious that the operators will vary from row to row, but for any one row (r = constant) they will be constant. It will be found however that block operators can still be constructed in an automatic manner. On the center line, $r = 0$, ϕ is usually prescribed, so that the possible difficulty arising from the $1/r$ term does not occur.

Biharmonic Operator. For equations involving the biharmonic operator, it is seen from equation (3–15) that the finite difference operators involve thirteen points. As such the problems are rather more difficult to relax. Block operators, and these are practically essential for obtaining solutions in finite time, cannot be written down by inspection, but must be built up as needed.

References for Chapter 3

(1) R. V. Southwell, *Relaxation Methods in Engineering Science* (Clarendon Press, Oxford, 1940); also *Relaxation Methods in Theoretical Physics* (1946).
(2) F. S. Shaw, *An Introduction to Relaxation Methods.* C.S.I.R. Div. of Aero. Report SM78 (Sept. 1946).
(3) F. S. Shaw, "The torsion of solid and hollow prisms in the elastic and plastic range by relaxation methods." Australian Council for Aero. Report ACA–11 (Nov. 1944).
(4) D. G. Christopherson and R. V. Southwell, "Relaxation methods applied to engineering problems. III. Problems involving two independent variables." Proc. Roy. Soc. **A168**, 317–350 (1938).

CHAPTER 4

THE QUEST FOR ACCURACY IN COMPUTATIONS USING FINITE DIFFERENCES

R. V. Southwell *

Perspective View of Relaxation Procedures. Ten years in which all my spare time has been given to the development of Relaxation Methods ** have yielded more than a list of problems solved; they have also afforded opportunities for observation of the differing modes of approach to computation which may be said to characterize mathematicians and engineers. I do not mean that no one whose training has been mathematical has, in regard to relaxation methods, preference for what I term the "engineer's approach"; or *vice versa*. But broadly speaking a training in mathematics seems to induce a liking for computation performed in accordance with fixed rules, and hence a tendency to look on the relaxational technique as *iterative;* whereas the engineer — in relaxation as in the workshop process of "hand-scraping" with which it has noteworthy similarity — seeks to retain the fullest measure of initiative, to place a premium on *intuition.**** I think there can be no doubt of the existence of two divergent points of view.

Let me say at once that, far from desiring uniformity, I find this divergence stimulating and desirable. It shows, as I like to think, that relaxation methods "have come to stay," — since otherwise men would not have strong views regarding the proper lines of their development; and it is good since it obliges both sides to examine the

* Rector, Imperial College, London, England.

** An abbreviation of the original (1935) title "Method of Systematic Relaxation of Constraints."

*** I cannot improve on the words of Professor Emmons: — "In fact, for the computer (as opposed to those who think only about the logic behind the computation methods) the relaxation method has a spirit lacking entirely from the iteration process. The former challenges one's intellect at each step to make the best possible guess, while the latter reduces one to the status of an automatic computing machine (without the advantage of no computational errors). It should not be inferred that the relaxation method *requires* high intellectual powers. If changes are chosen in a specifiable way, it reduces exactly to the iteration process. The computer can then vary from this completely specified process by whatever amount fits his own skill." (*Quarterly of Applied Mathematics*, Oct. 1944.)

grounds of their opinions. I make no claim to be myself impartial, for all of my own thinking has been along what I describe as "engineering" lines; but already I owe much to what I here term "mathematicians," and I expect to become still deeper in their debt as time goes on.

Finer Mesh *vs.* an "Improved" Finite Difference Approximation. There is another category of "relaxationists," — men whose main interest is to bring all processes of computation within range of some type of machine. I say nothing about them in this paper, which is confined to one particular field wherein the divergent modes of approach have been revealed: namely, *the quest for higher accuracy in methods which, as an initial step, substitute a finite difference approximation for the exact (partial differential) equation which governs the "wanted function" of a problem.*

Consider, for example, Poisson's equation in two independent variables, namely,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + Z(x,y) = 0 \qquad (4\text{–}1)$$

in which x and y are the independent variables, w is the "wanted function," and Z is some specified function of x and y. The simplest finite difference approximation to equation (4–1), normally used to compute w-values at nodes of a square-mesh net of mesh-side a, is

$$w_1 + w_2 + w_3 + w_4 - 4w_0 + a^2 Z_0 = 0 \quad (4\text{–}2)$$

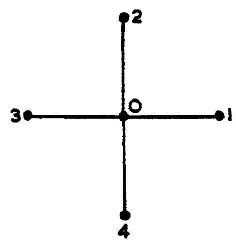

Fig. 4–1. Points Numbered for Use with Equation (4–2).

The suffixes 0, 1, 2, 3 and 4 relate to five typical points arranged as shown in Fig. 4–1. Both parties admit that the approximation of (4–2) to (4–1) becomes closer as the mesh-side a is reduced. But the "engineer's" reaction is to concentrate on finding means of rapid "advance to a finer net," whereas those whom I have described as mathematicians seem to prefer an alternative means to closer approximation: namely, instead of employing a finer net (which entails more labor) to use an "improved" approximation in the finite differences, — *by which is meant a relation differing from (4–2) by the inclusion of more nodal values.*

The case for this "mathematician's" line of approach has been ably presented in a recent paper by L. Fox (reference 1). Having remarked that equation (4–2) above in effect neglects a quantity which strictly speaking ought to appear on its right-hand side, he states three disadvantages of the engineer's alternative (namely, approach by stages

to an interval a so small that the neglected terms are in fact negligible): "First, a very great number of points have often to be taken in the relaxation process to ensure the vanishing of the difference correction, and the difficulty and labour of obtaining an accurate solution to the simultaneous equations increases rapidly with the number of equations involved. Secondly, derivatives of the computed function are often required in practical problems, and equations like (1) and (2) * show that the nth derivative depends for its accuracy on the nth difference, which gets small with the interval of tabulation. Thirdly, the criterion used initially for the legitimacy of neglect of difference corrections — namely, that relaxation on successive meshes gives the same result — has tended to be ignored in later applications."

Accelerating Convergence. Of these contentions the third must be admitted as true in fact, though the criterion is valid and *could* have been applied at any time; our energies have been devoted to extension of the *range* of relaxation methods, and when it was evident that attainment of closer accuracy was a matter merely of increased time and labor we have usually turned to other pioneering work. But in regard to the first and second, less apology is needed for the "engineer's approach." Admitting that the labor of solution must increase "with the number of equations involved" (or, as we prefer to say, with the fineness of the "ultimate net"), he may claim that the rapidity of this increase need not be prohibitive if, by exercise of ingenuity, special devices are employed to treat particular problems.

One such device has general application: the use of *graded nets* (reference 4, § 92 and § 93) to investigate fine detail in places where the wanted function varies rapidly. Another will now be given for the first time: a process of successive approximation for the solution of biharmonic problems (for example, flat plates).

Successive Approximation in Biharmonic Problems. The typical biharmonic equation is

$$\nabla^4 w = Z(x,y) \tag{4-3}$$

Z standing as before for some specified function of x and y, and ∇^2 denoting the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ which appeared in equation (4-1). To determine the solution of equation (4-3) we require a double boundary condition: that is, at every point on the boundary either w and its normal gradient may be specified (as is usual), or w

* These equation numbers are those of reference 1. The equations are given later with the numbers (4-9) and (4-10).

and $\nabla^2 w$ may be specified (as in the problem of flexure for a plate having "simply supported" edges).

The second of these cases is much the simpler, as was remarked in reference 2, § 1. For if

$$W + \nabla^2 w = 0 \tag{4-4}$$

then equation (4–3) can be written in the form

$$\nabla^2 W + Z(x,y) = 0 \tag{4-5}$$

from which W, *when its value is specified at every point on the boundary,* can be found by an application of the "plane-harmonic" relaxational technique; and knowing W we can deduce w from equation (4–4) — boundary values of w being specified — by another application of the same technique.

Technique with Boundary Values of w. The case is harder whenever w and its normal gradient have specified values on its boundary; then the more complicated "biharmonic technique" must be employed, and serious labor is entailed by the employment of an adequately fine net. But here (it may be contended by those who prefer the "engineer's approach") is the place at which to introduce a special device; and the following seems to satisfy requirements:

(1) Using a coarse net as suggested by Fox, *but retaining the simplest form of the finite difference approximation* (which Fox would desire to replace), use one or other standard form of the biharmonic technique to obtain a "first solution" to equation (4–3), *and from this deduce boundary values of* $\nabla^2 w$.

(2) Adopting these boundary values of $\nabla^2 w = -Z$, *on a fine net* use the simpler plane-harmonic technique, first to solve equation (4–5) and so find W everywhere, then to find w as the solution (it may be termed the "second solution") of equation (4–4).

(3) From w according to the "second solution" (say, w_2) compute the gradient of w_2 at points where "strings" cut the boundary, also $\nabla^2 w_2$ and $\nabla^4 w_2$ at nodal points in the field of integration, and hence $Z_2(x, y)$ as defined by

$$\nabla^4 w_2 = Z_2(x,y) \tag{4-6}$$

Then, if Z_2 be deemed a sufficiently close approximation to the specified function Z in equation (4–5), and if the gradients of w_2 be deemed sufficiently close to the specified gradients of w, the problem may be regarded as solved. If not, continue as follows:

(4) Subtract equation (4–6) from (4–3) to obtain

$$\nabla^4 w' = Z'(x,y) \tag{4–7}$$

where

$$\left.\begin{aligned} w' &= w - w_2 \\ Z'(x,y) &= Z(x,y) - Z_2(x,y) \end{aligned}\right\} \tag{4–8}$$

and evaluate the boundary gradients of w' (which has a zero value at every point on the boundary). Then the problem remains of computing the supplementary displacement w', which is governed by equation (4–7) and of which the boundary values and gradients (along with Z') are specified. This problem is thus similar in kind to that with which we started, *but it needs less exact treatment* because $Z'(x,y)$ — being an "error" — is small in relation to $Z(x,y)$, also $w' = 0$ on the boundary and its boundary gradient — being also an "error" — is everywhere small.

The essence of the foregoing device is its reduction of a biharmonic to a plane-harmonic problem which (being much simpler) *can without excessive labor be treated on a much finer net.* The case for it (from the "engineer's standpoint" adopted here) is that *no firm ground for certainty of sufficient accuracy exists except a conviction (admittedly intuitive) that the "ultimate net" is fine enough to render negligible the difference between the "net" and the "membrane analogue"* (reference 4, § 36 to § 38).

Investigation of Approximations

Basic Assumptions in the Calculus of Finite Differences. We turn now to the second of Fox's three contentions (reference 1, § 2), that equations like his equations (1) and (2), viz —*.

$$hf_0' = \delta_0' - \tfrac{1}{6}\delta_0''' + \tfrac{1}{30}\delta_0^{\text{v}} - \cdots \tag{4–9}$$

$$h^2 f_0'' = \delta_0'' - \tfrac{1}{12}\delta_0^{\text{iv}} + \tfrac{1}{90}\delta_0^{\text{vi}} - \cdots \tag{4–10}$$

— "*show* that the nth derivative depends for its accuracy on the nth difference, which gets small with the interval of tabulation." The italics have been inserted because my purpose is to question an assumption which is fundamental to that argument.

That assumption is the basis of the calculus of finite differences: that the "wanted" (and hence unknown) function of a problem can be approximately represented by a polynomial, *and that the approximation improves with the order of the polynomial.* As applied to tabulation of the mathematical functions (a field with which some part of Fox's

* Here δ_0^n is the nth central difference of $f(x)$ at $x = 0$.

paper deals) the assumption is often justified *ab initio* because the nature (as distinct from the numerical values) of the function is known throughout the range. But when no such knowledge exists (and this is normally the fact in plane-harmonic problems — to say nothing of more difficult problems to which relaxation methods have been applied) *I maintain that the polynomial assumption is open to question, and with it many accepted formulas in the calculus of finite differences.*

Illustrative Examples. Here are two examples which at least make this contention plausible. The first is an example of polynomial representation, the second an example of numerical integration. Both indicate that incorrect conclusions may result from the assumption that the approximation improves with the order of the polynomial.

Example 1. The function

$$y = \frac{1}{1 + x^2} \tag{4-11}$$

has values as below within the range $-1 \leq x \leq 4$:

$x =$	-1	0	1	2	3	4
$y =$	0.5	1.0	0.5	0.2	0.1	$\frac{1}{17}$

Making the polynomial assumption, and successively increasing the so-called "order of approximation," we represent y in turn by

$$y_1 = 1 - 0.5x^2 \tag{4-12a}$$

which gives y its correct values for $x = \pm 1$ and 0; by

$$y_2 = 1 - 0.2x - 0.5x^2 + 0.2x^3 \tag{4-12b}$$

which gives y its correct values for $x = \pm 1$, 0 and 2; and by

$$y_3 = 1 - 0.3x - 0.45x^2 + 0.3x^3 - 0.05x^4 \tag{4-12c}$$

which gives y its correct values for $x = \pm 1$, 0, 2 and 3. The curves of y and of its polynomial approximations y_1, y_2, y_3 are shown in Fig. 4-2, and while y_2 and y_3 are almost indistinguishable from y in the ranges $0 \leq x \leq 1$ and $1 \leq x \leq 2$ respectively, it will be seen that *in the range* $-1 \leq x \leq 0$ *the approximation of* y_1, y_2, y_3 *to* y decreases *in that order, that is, as the order of the representing polynomial is increased.*

In particular the values for ($x = 0$) of the first and higher derivatives of y deserve attention. The correct values are

$$(y')_0 = 0 \qquad (y'')_0 = -2 \qquad (y''')_0 = 0$$

whereas according to expressions (4–12, *a*, *b* and *c*)

$$
\begin{array}{lll}
(y_1')_0 = 0 & (y_1'')_0 = -1 & (y_1''')_0 = 0 \\
(y_2')_0 = -0.2 & (y_2'')_0 = -1 & (y_2''')_0 = 1.2 \\
(y_3')_0 = -0.3 & (y_3'')_0 = -0.9 & (y_3''')_0 = 1.8
\end{array}
$$

Thus only the first approximation gives y_0' *and* y_0''' *correctly; and all three approximations underestimate* y_0'' by 50 per cent.

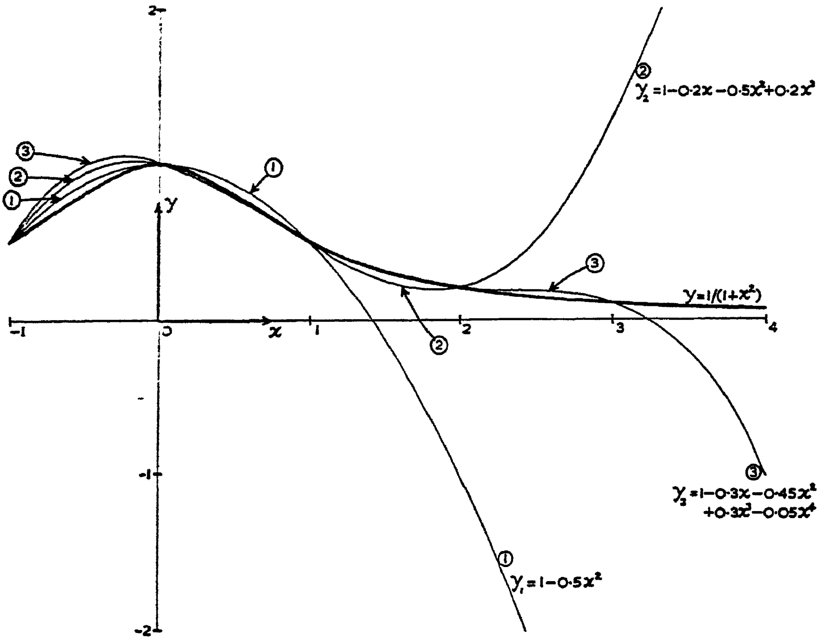

FIG. 4–2. CURVE OF $y = 1/(1+x^2)$ AND OF THREE APPROXIMATIONS OF DIFFERENT ORDERS.

Example 2. The value of the definite integral

$$4\int_0^1 \frac{dx}{1+x^2} \text{ is } 4[\tan^{-1} x]_0^1 = \pi = 3.14159265 \cdots$$

To compute the integral by the calculus of finite differences, we divide the range $0 \leq x \leq 1$ into any number of equal parts (or intervals) and we employ a formula of approximate integration based on some polynomial representation of the integrand. A different formula corresponds with each different order of polynomial, and if the accuracy of the representation increased with the order of the polynomial, so also would the accuracy of the numerical integration. But (as remarked by Karman and Biot in reference 3, § 9) the "four-strip

formula of integration" (which implies representation of the integrand by a polynomial of the fourth order) gives to the integral the value 3.14212; whereas "Simpson's rule" (which implies representation by a distinct quadratic for each pair of intervals, and hence *by two different quadratics in the range considered*) gives to the integral the more accurate value 3.14157.

Deductions. The deduction to be drawn from these examples seems to be, that *increased accuracy does not necessarily accompany an increased order of the polynomial used to represent a function.* It cuts across more than Fox's argument, for almost all of the calculus of finite differences rests on the "polynomial assumption"; yet I think it is not really paradoxical, considering the criterion that has been employed to assess the "error" of a representation. This is *the order of the lowest derivative in the error expressed as a Taylor series.*

Does not the adoption of this criterion amount to an assumption that the Taylor series is convergent throughout the whole range of the polynomial representation? And is it not unsafe (in general) to make the assumption — and even less safe to base on it a judgment of computed values? Of course, once polynomial representation is accepted, its consequences (which include such judgments) must be accepted too. But what are the grounds of acceptance? I find no more than this in Whittaker and Robinson (reference 5, § 8): "The problem of finding a 'smooth' curve to pass through the points A, B, C, D, . . . has not a unique solution; in fact an infinite number of curves satisfying these conditions can be found. As our aim is a practical one, *we naturally choose the simplest solution of our problem. Remembering that the simplest functions are polynomials,* we enquire . . . etc." (The italics have been added.)

Necessity for Employment of Fine Nets. As I believe, the conclusion to be drawn from the examples given is that *accuracy is not predictable of quantities computed from a finite difference approximation unless the interval is small enough to justify belief in the convergence of the Taylor series;* that in general the radius of convergence, though it exceeds one or two of the smallest intervals that are practicable, will not exceed many such intervals; and that consequently *what have been claimed as closer approximations may in fact be less accurate.* This is a justification of the "engineer's approach," in which the only basis of acceptance is a belief that fine enough intervals, or "nets," have been employed; and it is not permissible to oppose to it examples in which use of a higher order polynomial leads to more accurate results,

if in those examples the wanted quantities are functions in regard to which convergence of the Taylor series is predictable.

Another point may be made in passing. Although their accuracy is sufficient, results computed on a coarse net may fail to provide the detail which is needed for an accurate construction of contours. It is only in constructing contours (a process which entails plotting of the wanted quantity on lines parallel to x or y) that a sound — though intuitive — assessment of convergence can be made.

Conclusion. To summarize the contentions of this chapter: Any approximation in finite differences to a differential equation in one or more independent variables must entail an error which depends (1) upon the order of the differences, (2) upon the smallness of the intervals. In seeking to reduce this error, workers in the field of computation adopt (speaking generally) a different standpoint according as their approach is what I term "mathematical" or "engineering": (*a*) the "mathematician" tends to look for better approximation to a use of formulas containing differences of higher order; whereas (*b*) the "engineer" sees only one safe route to close approximation — namely, employment of sufficiently small intervals.

I have myself the "engineering" outlook (*b*), and in this discussion I defend it on the ground that (*a*) rests on an underlying assumption (the validity of "polynomial representation") which itself is open to doubt unless the interval is small. Examples in support of this contention are given, and the deductions to be drawn from them are discussed. I conclude that only by adopting small intervals can the computer have real confidence in his result; and I illustrate an alternative approach to the problem of saving labor, by one device which facilitates "advance to a finer net".

References for Chapter 4

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(2) L. Fox and R. V. Southwell, Phil. Trans. **C1**, 15–56 (1941); **A239**, 419–460 (1945).
(3) Th. Karman and M. A. Biot, *Mathematical Methods in Engineering* (McGraw-Hill, 1940).
(4) R. V. Southwell, *Relaxation Methods in Theoretical Physics* (Clarendon Press, Oxford, 1946).
(5) E. T. Whittaker and G. Robinson, *The Calculus of Observations* (2d ed.; Blackie and Son, 1926).
(6) D. N. de Q. Allen, L. Fox, H. Motz, and R. V. Southwell, Phil. Trans. **C1**, 85–97 (1941); **A239**, 488–500 (1945).

CHAPTER 5

A NEW APPROACH TO THE NUMERICAL SOLUTION OF LAPLACE'S EQUATION *

M. M. Frocht **

Synopsis. This paper deals with certain simplifications in the numerical solution of Laplace's Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \tag{5-1}$$

which markedly reduce the time and labor required to attain approximate convergence. The new approach consists of a simple and rapid method for the determination of *good initial values* which are equally useful in the method of iteration and in the method of relaxation. In many cases the initial values are sufficiently good to provide an approximate solution in themselves, without further improvement. *The procedure depends upon known boundary values.*

The method for obtaining the initial interior values consists of drawing several straight lines through the point at which the value of the harmonic function is required, these lines extending in opposite directions to the nearest boundaries and forming a rosette. Each line is viewed as a taut string passing above or below the given point and having a slope determined by the known values of the function at the two points of intersection of the straight line with the boundary. The ordinate under each string at the point in question is determined graphically and the arithmetic mean of all these ordinates in the rosette is taken as the initial value at the given point. The method described is called the *Linear Rosette Method.*

By combining the method of linear rosettes with the expressions for the key values in rectangles, an extremely rapid procedure is available for the determination of the values of the harmonic function not only

* This discussion is based on Chapter 8 of the author's *Photoelasticity* (Wiley, 1948), Vol. II. The figures in this chapter are reproduced with the permission of the publishers. For a general discussion of modern iteration methods the reader is referred to Chapters 8 and 9 of this book.

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at isolated points, but also along complete sections. The method is illustrated with numerical examples taken from stress analysis by means of photoelasticity.

INTRODUCTION

Basic Equations in Iteration and Relaxation. The place of Laplace's equation in technology is well known. It enters into problems of heat transfer, electrical potential, deflections of membranes, stress analysis, etc. Numerical methods have been found to be the most effective for the solution of Laplace's equation for *given boundary values* of a general harmonic function $U(x,y)$.

One of the effective procedures in the numerical solution is the so-called iterative process. The basic equation in this process is the four-point influence equation *

$$\left(\frac{1}{ac}+\frac{1}{bd}\right)u_o=\frac{u_a}{a(a+c)}+\frac{u_b}{b(b+d)}+\frac{u_c}{c(c+a)}+\frac{u_d}{d(d+b)} \quad (5\text{–}2)$$

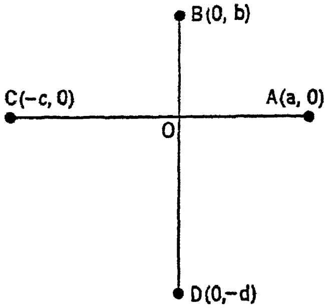

FIG. 5–1. DESIGNATION OF POINTS.

in which a, b, c, and d are as shown in Fig. 5–1. For equally spaced points equation (5–2) reduces to the Liebmann formula

$$4u_o = u_a + u_b + u_c + u_d \quad (5\text{–}3)$$

In the method of relaxation the Liebmann equation is written as

$$u_a + u_b + u_c + u_d - 4u_o = Q \quad (5\text{–}4)$$

in which Q is spoken of as the *residual* at the point considered.

Both iteration and relaxation employ processes of successive improvement of arbitrary initial values. In this paper it is assumed that the reader is familiar with these methods.

Considerable progress has been made in recent years in reducing the time and labor required to obtain a numerical solution by the iterative processes. This progress is due in the main to the development of special expressions for key values for simple rectangles, the employment of these key values in block iteration, and the introduction of the method of differences.** However, little attention has been paid

* For a derivation of this equation see the author's *Photoelasticity* (Wiley, 1948), Vol. II, p. 289.

** *Ibid.*, p. 276; or G. H. Shortley and R. Weller, "The Numerical Solution of Laplace's Equation," J. App. Phys., 9, 334 (1938).

to the initial values, and both the iterative and relaxation processes generally start with arbitrary initial values, usually zeros.

New Approach. It has long been the writer's belief that further savings in labor in the numerical solution of Laplace's equation can be affected by judiciously choosing the *initial values*, and in 1940, the author jointly with M. M. Leven published a method for the determination of such initial values. This method was based on equation (5–2) and involved the determination of the average ordinate of an area surrounding a point.*

The present paper is a further attempt at a rational approach to the numerical solution of Laplace's equation. A new and much simpler procedure is proposed for the initial values, which gives remarkably good approximations at critical points. By combining the proposed method with a simple expression for the key value in a (4×4) square, some rather interesting results can be obtained for sections of symmetry such as might occur in tension bars with circular holes or grooves.

Definition of Key Values. Let U denote a plane harmonic function defined for a region R by its values on the boundary T. A set of interior values of U so located in the region R to which the function U applies that, from these values, the values of U at all other network points can be calculated by the Liebmann or four-point formula

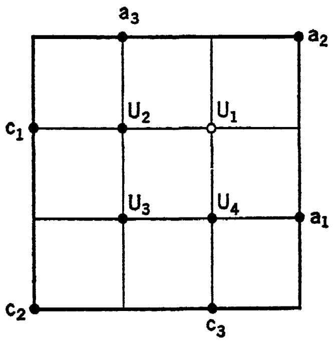

FIG. 5–2. 3×3 SQUARE.

applied either normally or diagonally, is defined as a set of *key values*. An interior, or unknown, value will always be denoted by u with a suitable subscript relating it to a given point. Known boundary values will be denoted by $a, b, c, \cdots$, etc. In a square (3×3) Fig. 5–2, any one of the four interior values u_1, u_2, u_3, u_4, is a key value for the

* "A Rational Approach to the Numerical Solution of Laplace's Equation," J. App. Phys., **12**, 596–604 (August, 1941).

whole region. Thus, if u_1 be known, u_3 can be found from the diagonal neighbors and u_2, u_4, from the normal neighbors.

Similarly in a rectangle (4×3), Fig. 5–3, either u_1 or u_4, is a key value for the whole region. Thus if u_1 be known, u_3 and u_5 can be found from diagonal neighbors and u_2, u_4, u_6 from normal neighbors. A similar condition exists in a (4×4) square, Fig. 5–4, in which u_1, the value at the center of the square, defines the remaining eight interior values. Four of these values, u_2, u_3, u_4, u_5, are found from diagonal neighbors, and the remaining four from normal neighbors. In all three cases, one key value is sufficient to obtain a complete numerical solution of Laplace's equation.

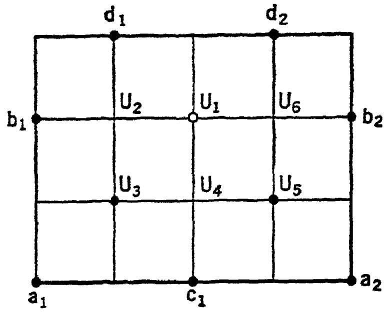

FIG. 5–3. 4 × 3 RECTANGLE.

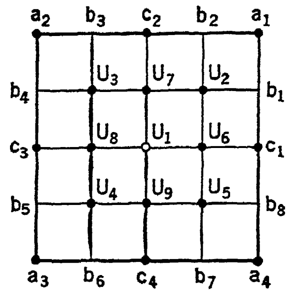

FIG. 5–4. 4 × 4 SQUARE.

Key Value for a (4 × 4) Square, Fig. 5–4. The simplest procedure to obtain an expression for the key value in a (4×4) square is to express u_1 in terms of u_2, u_3, u_4, and u_5, and then to express these quantities in terms of u_1. Using equation (5–3) we obtain the following five expressions:

$$\left.\begin{aligned}
4u_1 &= u_2 + u_3 + u_4 + u_5 && (a)\\
4u_2 &= u_1 + a_1 + c_1 + c_2 && (b)\\
4u_3 &= u_1 + a_2 + c_2 + c_3 && (c)\\
4u_4 &= u_1 + a_3 + c_3 + c_4 && (d)\\
4u_5 &= u_1 + a_4 + c_4 + c_1 && (e)
\end{aligned}\right\} \quad (5\text{–}5)$$

Upon solution these yield

$$6u_1 = 0.5 \sum_{i=1}^{4} a_i + \sum_{i=1}^{4} c_i \qquad (5\text{–}6)$$

Initial Values by the Linear Rosette Method. The values of U define a free surface under uniform tension which may be roughly approximated by assuming threads or strings to pass over the given point in different directions to the terminal values on the boundary.

These threads may be thought of as fixed at one end and under constant tension at the other. Each thread is in general raised or lowered by an intersecting one. Ultimately they concur at one point, the height of which above the datum plane is, as a first approximation, equal to the arithmetic mean of the heights of all the strings passing over that point. Referring to Fig. 5–5(*a*), $(u_o)_1$ is one linear approximation to u_o from the string A–A', corresponding to the boundary values u_a and $u_{a'}$, at points A and A' respectively. Such linear values can be found graphically in a few minutes, and they will always be of great help. The graphical evaluation of initial values is somewhat facilitated by laying off the end values u_a and $u_{a'}$ from a fixed base line on a piece of transparent graph paper, Fig. 5–5(*b*).

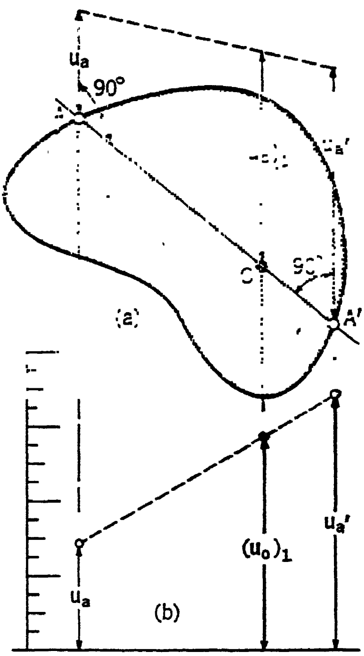

FIG. 5–5. STRING FROM BOUNDARY VALUE u_a TO BOUNDARY VALUE u_a' DETERMINES $(u_0)_1$, A FIRST APPROXIMATION OF U_0.

This method of determining initial values will be referred to as the linear rosette method. The number of lines in the rosette will depend on the manner in which the boundary values vary. The sharper this variation, the greater should be the number of lines or strings.

Example. As an illustration of the significance of initial values and of the rosette method we will solve the problem of the (3×3) square shown in Fig. 5–2. Using an equiangular rosette and boundary values from Fig. 5–6, we find $(u_1)_0 = 24.5$ or roughly 24. From diagonal values we obtain

$$(u_2)_0 = \tfrac{1}{4}(16 + 48 + 72 + 24) = 40$$

Using normal points we have

$$(u_3)_0 = \tfrac{1}{4}(16 + 24 + 40) = 20$$

and

$$(u_4)_0 = \tfrac{1}{4}(40 + 72 + 64 + 24) = 50$$

These are our initial values. One traverse gives

$$u_1 = 25.5, \quad u_2 = 41.5, \quad u_3 = 20.75, \quad u_4 = 50.75$$

and the next improvement gives

$$u_1 = 25.9, \quad u_2 = 41.9, \quad u_3 = 20.95, \quad u_4 = 50.95$$

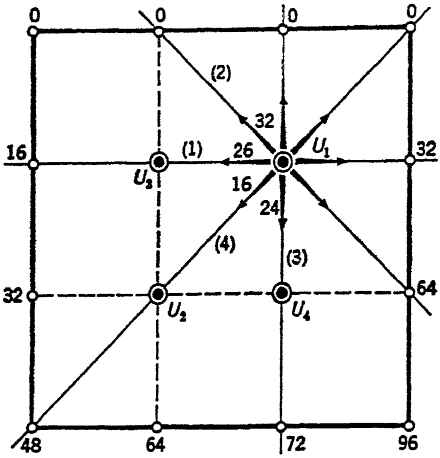

FIG. 5–6. SKETCH OF AN EQUIANGULAR LINEAR ROSETTE FOR THE EVALUATION OF U_1.

It can be shown that in order to arrive at essentially the same values from initial zero values it would require five traverses. We have thus accomplished in two traverses essentially the same results which were previously accomplished in five.

APPLICATION TO PHOTOELASTICITY

Laplace's Equation in Two-Dimensional Photoelasticity. Photoelasticians are interested in Laplace's equation because it furnishes an effective method of determining the sums of the principal stresses at a point in two-dimensional problems from direct photoelastic data. Thus, it is well known that in two-dimensional problems the sum Σ of the principal stresses p and q satisfies Laplace's equation, that is,

$$\left.\begin{aligned} &\frac{\partial^2\Sigma}{\partial x^2} + \frac{\partial^2\Sigma}{\partial y^2} = 0 && (a) \\ &\text{where} \\ &\Sigma = p + q && (b) \end{aligned}\right\} \quad (5\text{–}7)$$

It may be assumed that it is possible to obtain photoelastic stress patterns which yield boundary values of Σ of high accuracy. Hence, by Dirichlet's theorem the values of Σ are uniquely determined at every interior point of the stressed two-dimensional model.

The Linear Rosette and the Lateral Extensometer. In photoelasticity it is often necessary to find the sum of the principal stresses

at an isolated point. The sum is then combined with the difference of the principal stresses, which is given by the photoelastic patterns, to determine the principal stresses themselves.

One of the basic methods to determine the sum of the principal stresses $(p + q)$ rests on the equation

$$\delta = -\frac{\mu t(p + q)}{E} \qquad (5\text{-}8)$$

in which δ is the lateral deformation, μ is Poisson's ratio, E is the modulus of elasticity and t is the thickness of the model. From the meas-

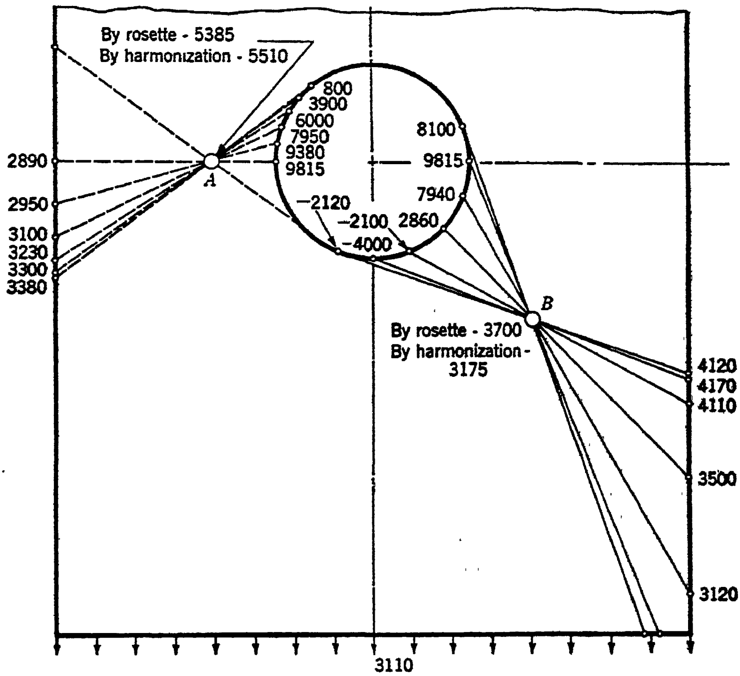

FIG. 5-7. LINEAR ROSETTES FOR U_a AND U_b.

ured deformation δ and the known values of μ, E, and t, the sum $(p + q)$ can be calculated. The instruments used to measure δ are known as lateral extensometers. These are precision instruments and must be capable of measuring quantities of the order of magnitude of 0.000001 in.

When a good lateral extensometer is not readily available, the linear rosette may be used as an approximate substitute. Experience shows that it generally yields good approximations, as can be seen from Fig. 5-7. The true values of $(p + q)$ obtained by iteration (har-

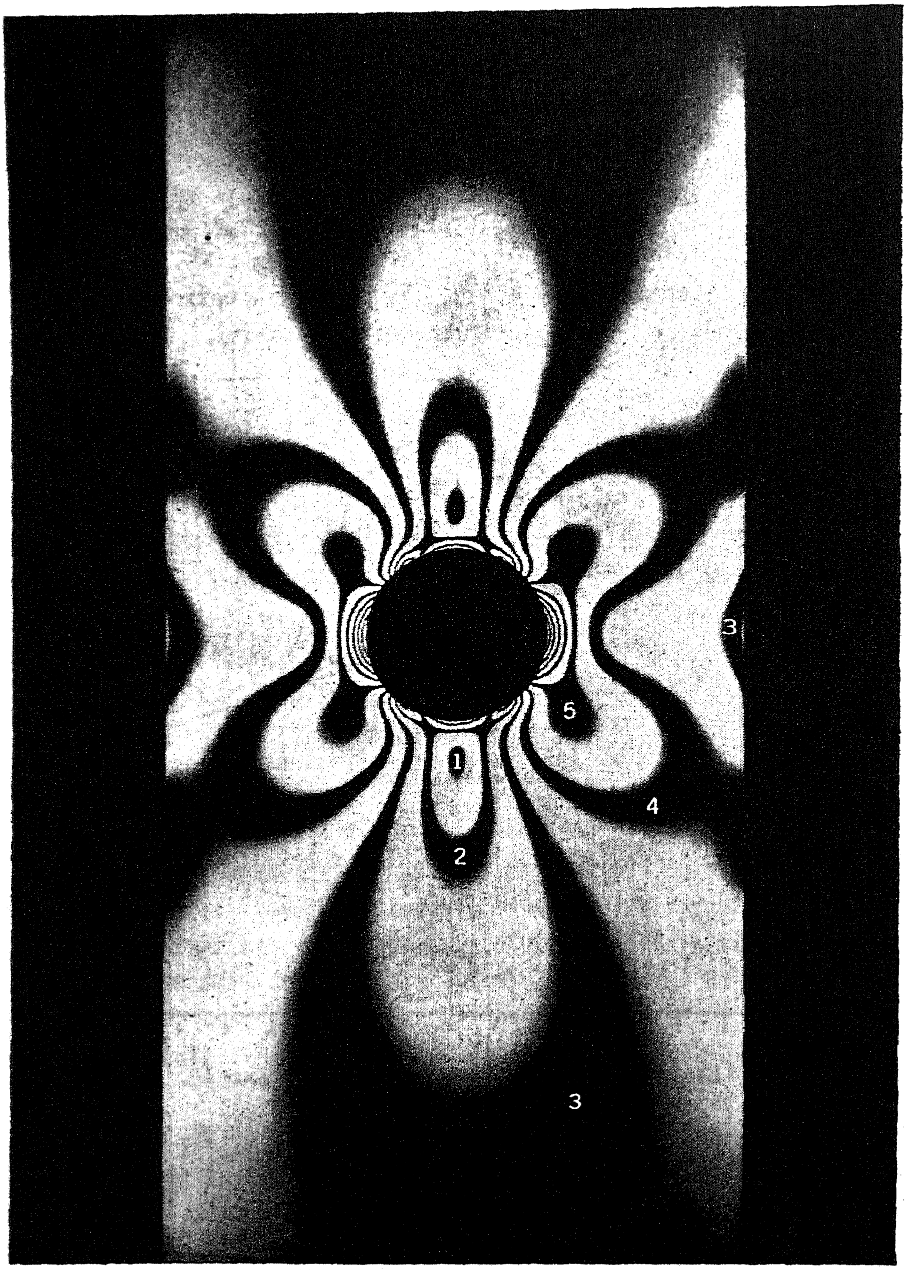

Fig. 5–8. Phoelastic Pattern from which the Boundary Values of Fig. 5–7 Were Obtained.

monized values) are 5510 and 3175 at A and B respectively. The corresponding values from the rosettes are 5385 and 3700. Figure 5–8 shows the stress pattern from which the boundary values were taken.

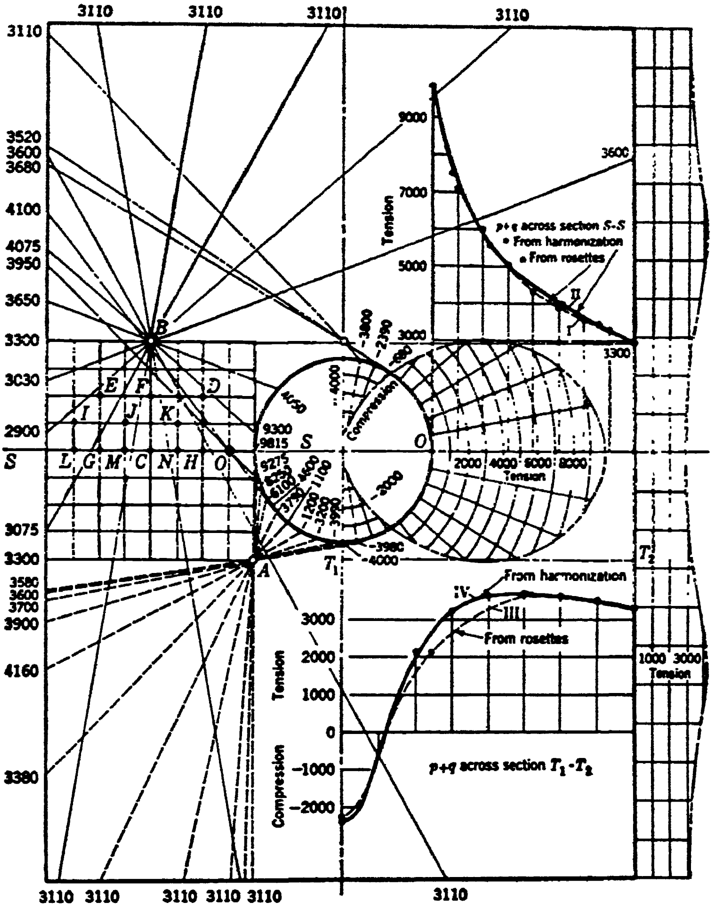

FIG. 5–9. DRAWING SHOWING THE APPLICATION OF THE LINEAR ROSETTE IN COMBINATION WITH EQ. 5–6 FOR A (4 × 4) SQUARE TO THE RAPID EVALUATION OF APPROXIMATE PRINCIPAL STRESSES IN A BAR WITH A HOLE IN TENSION.

Results for the section of symmetry $S-S$ are given by curve I, and for section T_1-T_2 by curve III. The harmonized values may be viewed as the limiting or convergent values obtained by iteration.

Near a source of stress concentration the type of rosette shown, in which the limiting strings in the rosette are tangent to the discontinuity, has been found to be quite sufficient. *The accuracy seems to increase as the point gets closer to the discontinuity.*

Sections of Symmetry. The linear rosette can be combined with equation (5–6) for a (4 × 4) square to give a good approximation to

the true distribution of $(p + q)$ across sections of symmetry. Thus in Fig. 5-9 we find the complete distribution across the horizontal section of symmetry by determining $(p + q)$ at A and B from rosettes and then calculating the value at the center of the square by equation (5-6).

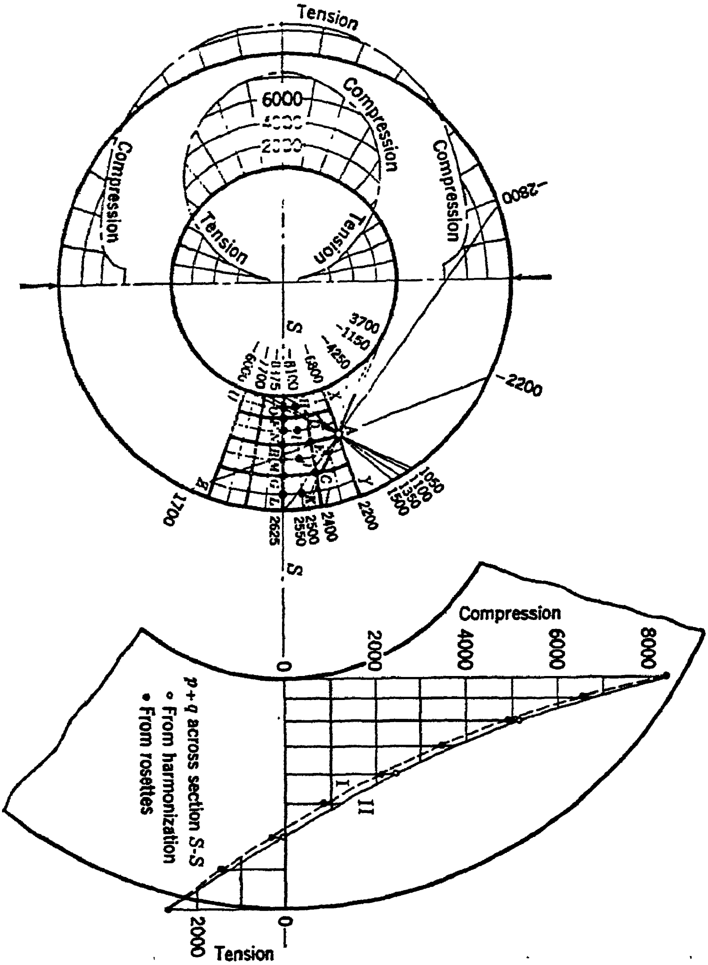

FIG. 5-10. DRAWING SHOWING THE APPLICATION OF THE LINEAR ROSETTE IN COMBINATION WITH EQUATION 5-6 FOR THE EVALUATION OF APPROXIMATE PRINCIPAL STRESSES IN A CIRCULAR RING.

The results are shown by curve I, Fig. 5-9. Curve II shows corresponding values from iteration. It is seen that the proposed method yields very good results.

General Section. Curve III, Fig. 5-9, gives the values of $(p + q)$ across section $T_1 - T_2$. It is seen that even along this section the

results are satisfactory. Attention is called to the maximum value which was determined from a rosette, showing again that the accuracy of the rosette is particularly good near sources of stress concentrations.

Another example is shown in Fig. 5–10. It can be shown that the polar rectangle $XYZU$ transforms into a (4×4) square. In this case the value of $(p + q)$ at A has to be found by a rosette. The remaining six values which enter into equation (5–6) are known boundary values. The results may therefore be expected to be particularly good, and curves I and II show that they are. The method is applicable to any radial section.

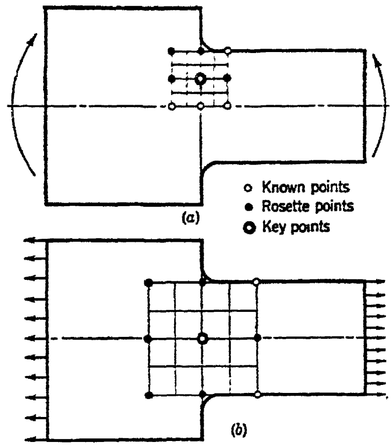

Fig. 5–11. Sketches Showing Possible Application of Linear Rosette and Equation 5–6 to Bars with Fillets.

Figure 5–11 shows how the same ideas can be applied to a bar with fillets in tension or bending. The method also gives good results in a tension bar with deep grooves.

Summary. A simple, rapid, graphical method is proposed to determine initial values for the numerical solution of Laplace's equation for a harmonic function with known boundary values.

Experience shows that the linear rosettes give good approximations to the true values at interior points and that the accuracy of the results is exceptionally good near boundaries in general and near stress concentrations in particular. The linear rosette may thus be used as an approximate lateral extensometer.

Further, by combining the initial values from the linear rosette with the expression for the key value in a (4×4) square, an effective method is available for the rapid determination of the principal stresses across sections of symmetry. Applications are also shown to bars with fillets in tension and bending.

Finally, for the general case, the initial values from linear rosettes will always reduce the labor and time necessary to arrive at approximate convergence, and for many engineering problems the initial values obtained from the linear rosettes provide satisfactory approximations without further improvement.

It is suggested that this method has a useful application to the problem of heat transfer in the steady state.

CHAPTER 6

NUMERICAL SOLUTIONS FOR THERMAL SYSTEMS

L. M. K. BOELTER * AND MYRON TRIBUS **

General Comments on the Solution of a Problem. The solution of a thermal problem may be obtained by either analytical or experimental means. The latter will not be discussed here. In the former, the system is idealized; equations representing the behavior are written and solved. As the idealization is improved by the introduction of additional elements, force fields, and variable parameters, the prediction of the behavior of the actual system is improved.

Generalizing, we may write

$$\lim_{n \to \infty} I_n \to \text{Behavior of actual system}$$

where n is the number of idealizations.

Often the behavior equations cannot be solved by classical means in terms of known and tabulated functions. Then numerical methods are resorted to.

But numerical (and graphical) methods are often more powerful than the algebraic means. Solutions involving variable properties of different boundary conditions may be accomplished by such procedures.

Machine methods allow the application of numerical methods to two and three dimensions; these are usually tedious when attempted by ordinary means. A schedule of operations is made for the problem solution, then the remaining solutions in this class of problems become automatic.

The idealizations must be re-examined at intervals. Experimental checks of the results are mandatory.

Numerical methods will yield solutions of a diverse series of systems described by appropriate differential equations and initial and boundary conditions. Through this means, knowledge of new functions

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(not yet studied and tabulated) and combinations of functions will be evaluated.

Numerical results obtained for particular systems may be generalized to be applicable to all members of that class of systems by deriving the appropriate dimensionless moduli for the system and the differential equation with its boundary conditions.

The ordering and assembling of these solutions is important. The process will be greatly facilitated provided authors will present all data on which solutions are based and include the results in tabular form.

Nomenclature and Symbols

Distance in feet		x
Error, ° F		E
Errors incurred by replacing $\frac{\partial t}{\partial x}$ by $\frac{\Delta t}{\Delta x}$	° F/ft	$E_{1,2}$
Error " " " $\frac{\partial^2 t}{\partial x^2}$ by $\frac{\Delta\left(\frac{\partial t}{dx}\right)}{\Delta x}$	° F/ft²	E_3
Error " " " $\frac{\partial t}{\partial \theta}$ by $\frac{\Delta t}{\Delta \theta}$	° F/hr	E_θ
Heat capacity at constant pressure, Btu/lb ° F		C_p
Symbol for finite difference		Δ
Number of arithmetical operations		p
Number of intervals of time $\Delta\theta$ in θ, i.e., $\theta = n\,\Delta\theta$		n
Reciprocal of the Fourier difference modulus (dimensionless)		$M = \frac{(\Delta x)^2}{a\,\Delta\theta}$
Subscripts denoting position		$a, b, c \cdots$
Subscripts denoting time		$1, 2, 3 \cdots$
Subscript indicating wood		w
Subscript indicating cork		c
Temperature, ° F		t
Thermal conductivity, Btu/hr ft² (° F/ft)		k
Thermal diffusivity, $\frac{k}{\gamma C_p}$, ft²/hr		a
Time in hours		θ
Unit thermal conductance, Btu/hr ft² ° F		h_c
Weight density, lb/ft³		γ
Weighting factors used at an interface. (Equations 6–6, 6–9, 6–10)		$\alpha, \beta, \epsilon, \delta$
Ambient air temperature		τ_∞

Rapid Numerical Solution of the Unidirectional Heat Conduction Equation. As an example of the application of analysis in terms of numerical and algebraic methods to the solution of a thermal system, consider the following problem statement: A refrigerator wall consists of 0.9 inch of cork and 1.0 inches of wood in intimate contact. The refrigerator and its surroundings are initially at 50° F. What will be the heat flow at the cork surface if this surface is lowered in temperature at a rate of five degrees per minute until the surface is at a temperature of −25° F and then held at this temperature? (See Fig. 6–5.)

The following idealizations are applied:

(1) The properties of the wood and cork are independent of direction, time, and temperature.
(2) There is no thermal resistance at the interface between cork and wood.
(3) The flow of heat is unidirectional.
(4) The properties, as measured by steady state means, are the same in the transient condition.

Mathematical Relationships. As a result of the above idealizations we may write an equation which describes the behavior of a system obeying the above postulates:

$$a\frac{\partial^2 t}{\partial x^2} = \frac{\partial t}{\partial \theta} \tag{6–1}$$

with the boundary conditions:

$$\begin{array}{lll} \text{at } \theta = 0 & t = 50 \text{ for all values of } x & \\ \quad\; x = 0 & t = 50 - 300\theta & 0 \le \theta \le \frac{1}{4} \\ & t = -25 & \theta \ge \frac{1}{4} \end{array}$$

$x = 0.075$ ft, all values of θ

$$k_w\left(\frac{\partial t_w}{\partial x}\right) = k_c\left(\frac{\partial t_c}{\partial x}\right)$$

$x = 0.1582$ ft, all values of θ

$$k_w\left(\frac{\partial t_w}{\partial x}\right) = h_c(t_w - \tau_\infty)$$

In the above equation and boundary conditions the terms have been defined under the heading *nomenclature*. However, we may especially note that

x = distance from cork surface, feet
h_c = unit thermal conductance between wooden (outside) surface of refrigerator and the ambient air, Btu/hr ft^2 °F
τ_∞ = ambient air temperature

Numerical Technique to Be Used. In this case, because of the awkward nature of the boundary conditions and the change in properties at the cork to wood interface, it is expedient to resort to a numerical technique for solving this problem. This technique consists of replacing the derivatives in equation (6–1) by ratios of finite differences. It is agreed at the outset that these differences must be small, but if the differences are too small, an inordinate amount of time will be required to attain a solution. A purpose of this paper is to examine the usual techniques for obtaining a solution and to suggest a method for achieving the most accurate solution for a given expenditure of labor. A by-product of this investigation is a method for solving the inverse problem, that is, given a desired temperature at an interior point of the solid, what must be the surface variations in temperature to achieve it? This latter consideration will be discussed first.

The Rapid Method

Schmidt-Binder Method. If one uses the so-called Schmidt-Binder method (reference 1, p. 103) one imposes upon the increments of time and space the restriction that

$$M = \frac{(\Delta x)^2}{a\,\Delta\theta} = 2 \tag{6–2}$$

and thereby obtains as a working equation:

$$t_{x,\theta+\Delta\theta} = \frac{t_{x-\Delta x,\theta} + t_{x+\Delta x,\theta}}{2} \tag{6–3}$$

The derivation of equation (6–3) is presented in reference 2. Reference 5 presents several alternate working equations. Equation (6–3) states that *the temperature at some point* x, *and at time $\theta + \Delta\theta$, is equal to the average of its neighboring temperatures at points* Δx *to either side and at the time θ.* A procedure to accomplish this *averaging process* will now be demonstrated which permits certain useful short cuts.

Work Sheets. Figure 6–1 shows a typical work sheet in which columns represent positions inside a solid and rows represent time. If the solid is homogeneous, the temperatures in each row are obtained

by an averaging procedure indicated by the arrows in Fig. 6–1. The temperatures at the tails of the arrows are averaged and the results written in the indicated squares. Thus each successive row follows from the row immediately above as explained in the caption to Fig. 6–1.

This method of computing the temperature at any particular row and column (time or position) depends only upon *the initial and boundary temperatures.* Since the work sheet represents the simultaneous

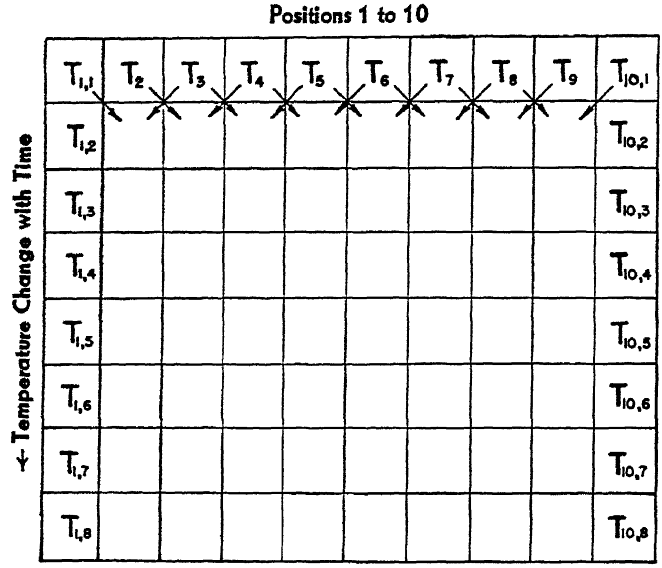


FIG. 6–1. WORK SHEET FOR HOMOGENEOUS SOLID.

Typical work sheet for numerical solution of Fourier-Poisson equation for unidirectional heat flow in a homogeneous solid. The temperatures at the tails of the arrows are averaged together arithmetically and the result written in the squares indicated. For example, T_3 and T_5 are averaged and written under T_4 in row 2 of the table. When row 2 is completed in this manner its values are then averaged in an identical manner to fill in row 3, and so on until all rows of the table are completed.

solution of linear algebraic equations, it should be possible in using this scheme to write an equation for the temperature at any point and at any time in terms of the given boundary values only. Reference 6 (p. 103) suggests this method for certain cases.

Consider Fig. 6–2, the work sheet for a homogeneous slab whose two faces are subject to temperature variations as indicated in the side columns and whose initial temperature distribution is given by the temperatures indicated in the first row. Suppose now it is desired to know the temperature at the time and position marked $t_{e,8}$ in Fig. 6–2.

Numerical Operations. Using the method of averaging temperatures presented in equation (6–3) the temperature at $t_{e,\,8}$ is computed as equal to the average of the temperatures at $t_{d,\,7}$ and $t_{f,\,7}$ in Fig. 6–2. These in turn are taken as equal to the averages of $t_{c,\,6}$, $t_{e,\,6}$ and $t_{e,\,6}$, $t_{g,\,6}$. That is:

$$t_{e,\,8} = \frac{t_{d,\,7} + t_{f,\,7}}{2} = \frac{t_{c,\,6} + 2t_{e,\,6} + t_{g,\,6}}{4} \tag{6–4}$$

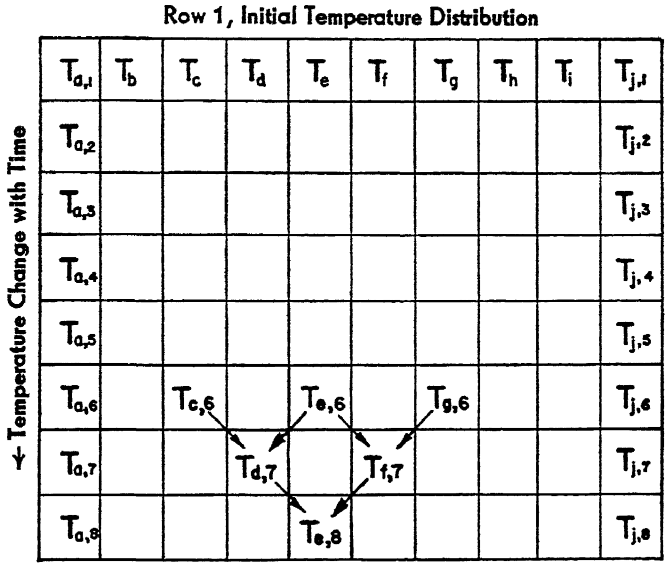


FIG. 6–2. WORK SHEET FOR HOMOGENEOUS SLAB.

The slab may be visualized as extending vertically with columns 1 and 10 representing its two surfaces. Initial temperature variation through its thickness is given by row 1. Successive rows give the temperature variation through the slab at successive times, θ_1 to θ_8. Lower values in the table are obtained from values in the row above by the averaging procedure, *i.e.*, the temperature $T_{e,8}$ is the average of $T_{d,7}$ and $T_{f,7}$. These in turn are obtained by averaging $T_{c,6}$ and $T_{e,6}$ and averaging $T_{e,6}$ and $T_{g,6}$.

To properly schedule the averaging method, the *zone of influence* is marked by arrows. That is, starting at $t_{e,\,8}$, as in Fig. 6–3, arrows are drawn tracing the paths along which the temperatures that determine $t_{e,\,8}$ exert their influence. In the center of each square the sum of the quantities at the tails of the two arrows leading into that square are written. At the extreme right is written 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$ $(\frac{1}{2})^n$ as shown in Fig. 6–3. Whenever the arrows are drawn into a square where the temperature is presumed known, no more arrows lead out of that square. By adding the terms that appear at the edge of the zone of

influence, and with due regard for the multipliers at the right, one obtains:

$$t_{e,8} = \tfrac{1}{16}t_{a,4} + \tfrac{1}{32}t_{j,3} + \tfrac{4}{64}t_{a,2} + \tfrac{1}{128}[14t_{b,1} + 34t_{d,1} + 35t_{f,1} + 20t_{h,1} + 5t_{j,1}] \quad (6\text{–}5)$$

The construction of the above matrix and solution for the indicated equation required less than two minutes. It will be noticed that equation (6–5) is explicit in $t_{e,8}$ and the effect of varying any one of the boundary temperatures can be determined directly.

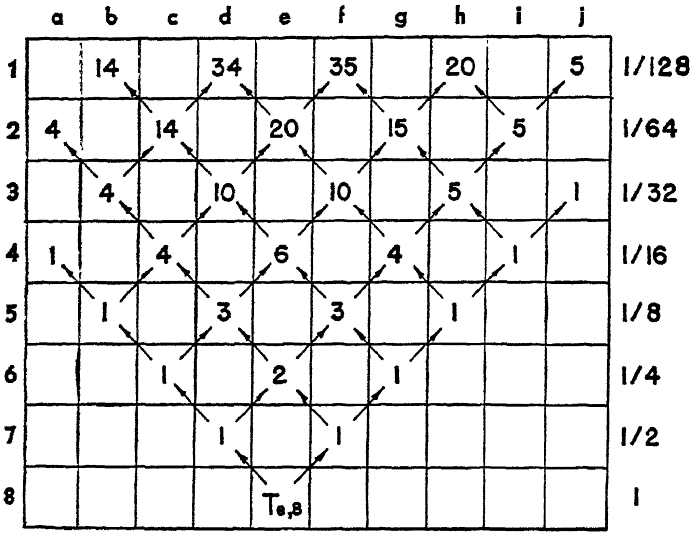


FIG. 6–3. ZONE OF INFLUENCE FOR $T_{e,8}$.

The "zone of influence" is constructed by drawing arrows from the square where the desired temperature is indicated, tracing the path whereby the boundary and initial temperatures insert their influence. At the right are the influence factors by which boundary values in corresponding rows are to be multiplied and added to determine $T_{e,8}$.

Extension to Nonhomogeneous Media and Convective Heat Transfer at the Surfaces. At the interface between two solids, equation (6–3) can no longer be used. It is possible, however, as shown in reference 3, to adopt an averaging process at an interface at which the contact resistance is zero in the form

$$t_{s,\theta+\Delta\theta} = \frac{\alpha}{2} t_{s-\Delta x',\theta} + \frac{\beta}{2} t_{s+\Delta x'',\theta} \quad (6\text{–}6)$$

Where the width of the laminae $\Delta x'$ and $\Delta x''$ are different in the two media and related to their respective thermal diffusivities by equation (6–7).

$$\frac{\Delta x'}{\Delta x''} = \sqrt{\frac{a'}{a''}} \quad (6\text{–}7)$$

Similarly, at a surface, the averaging method depends upon the thermal resistance to each side of the surface, and a weighting equation can be developed in the form of equation (6–8)

$$t_{s,\theta+\Delta\theta} = \frac{\alpha}{2}\tau_x + \frac{\beta}{2}t_{s+\Delta x,\theta} \tag{6–8}$$

Revised Work Sheet. Figure 6–4 shows a work sheet for this case in which the edge columns have heavier lines as a reminder that *these*

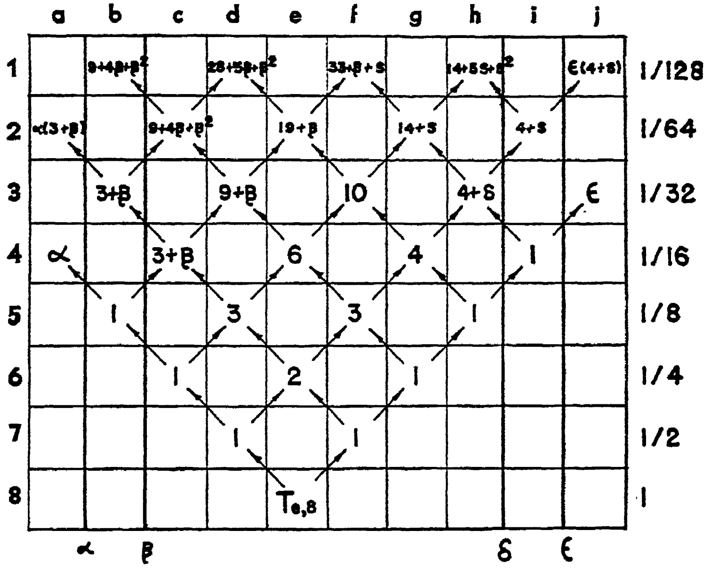


FIG. 6–4. SPECIAL AVERAGING PROCEDURE

The "zone of influence" for a nonhomogeneous solid or a solid where convection occurs at the surface must be constructed with proper regard for the unequal weighting of the temperatures to either side of the surface of discontinuity.

columns are averaged differently than the other columns, that is, equation (6–8) is to be used instead of equation (6–3). Note that whenever an arrow is drawn out of the column at the right or left edge its tail value is multiplied by α or β, but not when the arrow is drawn into the same column. This is because α and β equal unity for these neighboring columns. In Fig. 6–4 the working equations for the columns at the edge are:

$$t_b = \frac{\alpha}{2}t_a + \frac{\beta}{2}t_c \tag{6–9}$$

$$t_i = \frac{\delta}{2}t_h + \frac{\epsilon}{2}t_j \tag{6–10}$$

In equations (6–9) and (6–10), $\alpha + \beta = 2$, $\delta + \epsilon = 2$. The value of $t_{\epsilon,8}$ is now seen to be

$$
\begin{aligned}
t_{\epsilon,8} = {} & \tfrac{1}{16}\alpha t_{a_4} + \tfrac{1}{32}\epsilon t_{j_2} + \tfrac{1}{64}[\alpha(3 + \beta)t_{a_2}] + \tfrac{1}{128}[(9 + 4\beta + \beta^2)t_b] \\
& + \tfrac{1}{128}[28(3 + \beta^2)t_d + (33 + \delta + \beta)t_f \\
& + \{(14 + \delta) + \delta(4 + \delta)\}t_h + (4 + \delta)\epsilon t_j] \qquad (6\text{–}11)
\end{aligned}
$$

The equation above, of course, would reduce to a more compact equation if numerical values were used in place of α, β, and ϵ. As an example of the use of this method, the following problem has been prepared.

Sample Calculation Using the Techniques Presented

Statement of the Problem. Consider the problem of the heat transfer from the composite cork-wood refrigerator wall discussed earlier in this chapter (Fig. 6–5). By the use of appropriate idealizations the system may be described as obeying the Fourier-Poisson equation:

$$a\frac{\partial^2 t}{\partial x^2} = \frac{\partial t}{\partial \theta}$$

with boundary condition:

$$
\begin{array}{lll}
\theta = 0 & t(x,\theta) = +50^\circ\text{ F} & \\
x = 0 & t(0,\theta) = 50 - 300\theta & 0 \le \theta \le \frac{1}{4}\text{ hour} \\
 & t(0,\theta) = -25 & \frac{1}{4} \le \theta
\end{array}
$$

$$x = 0.075\text{ ft, (all } \theta) \qquad k_w\frac{\partial t_w}{\partial x} = k_c\frac{\partial t_s}{\partial x}$$

$$x = 0.1582\text{ ft, (all } \theta) \qquad k_w\left(\frac{\partial t_w}{\partial x}\right) = h_c(t_w - \tau_\infty)$$

Postulate that $h_c = 2$ Btu/hr ft² °F at the outer surface ($x = 0.1582$ ft). Calculate the heat flow at $x = 0$ from time $\theta = 0$ to $\theta = 40$ minutes.

Solution. In the construction of the work sheet the following data were used:

$$
\begin{aligned}
\Delta\theta &= 1/30\text{ hr} \\
\Delta x &= 0.223\text{ in. for wood} \\
\Delta x &= 0.201\text{ in. for cork} \\
k &= 0.025\text{ Btu/hr ft}^2\text{ (}^\circ\text{ F/ft) for cork} \\
k &= 0.12\text{ Btu/hr ft}^2\text{ (}^\circ\text{ F/ft) for wood} \\
\gamma &= 10\text{ lb/ft}^3\text{ for cork} \\
\gamma &= 50\text{ lb/ft}^3\text{ for wood}
\end{aligned}
$$

$$c_p = 0.485 \text{ Btu/lb } ^\circ\text{F for cork}$$
$$c_p = 0.57 \text{ Btu/lb } ^\circ\text{F for wood}$$
$$a = 5.16 \times 10^{-3} \text{ ft}^2/\text{hr for cork}$$
$$a = 4.21 \times 10^{-3} \text{ ft}^2/\text{hr for wood}$$

The heat flow is given by

$$q = -k\left(\frac{\partial t}{\partial x}\right)_{x=0} \cong k\frac{\Delta t}{\Delta x} = k\frac{t_1 - t_0}{\Delta x}$$

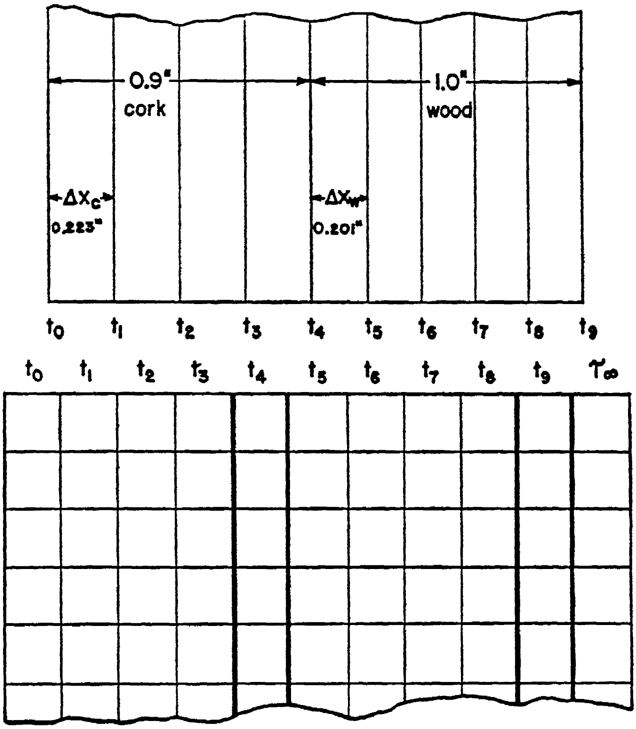


FIG. 6–5. SAMPLE PROBLEM FOR ILLUSTRATING THE TECHNIQUE

A nonhomogeneous wall consisting of 0.9″ of cork with 1.0″ of wood. The wood and cork are considered as divided into laminae of 0.223 and 0.201 inches respectively. The temperatures at the edges of these laminae are entered in the columns of the work sheet shown below the wall section. The column marked t_4 is drawn with a heavy border to remind the computer that the temperature in this column (which is the temperature at the wood-cork boundary) is obtained by a different averaging procedure than the other columns. t_9 is treated similarly.

Since $t(0,\theta)$ is given, it is only necessary to calculate $t(1,\theta)$ as a function of time. Figure 6–5 shows the sketch of the cork-wood wall and below it the matrix used as a working form. The temperatures are computed from the following relationships (see p. 98).

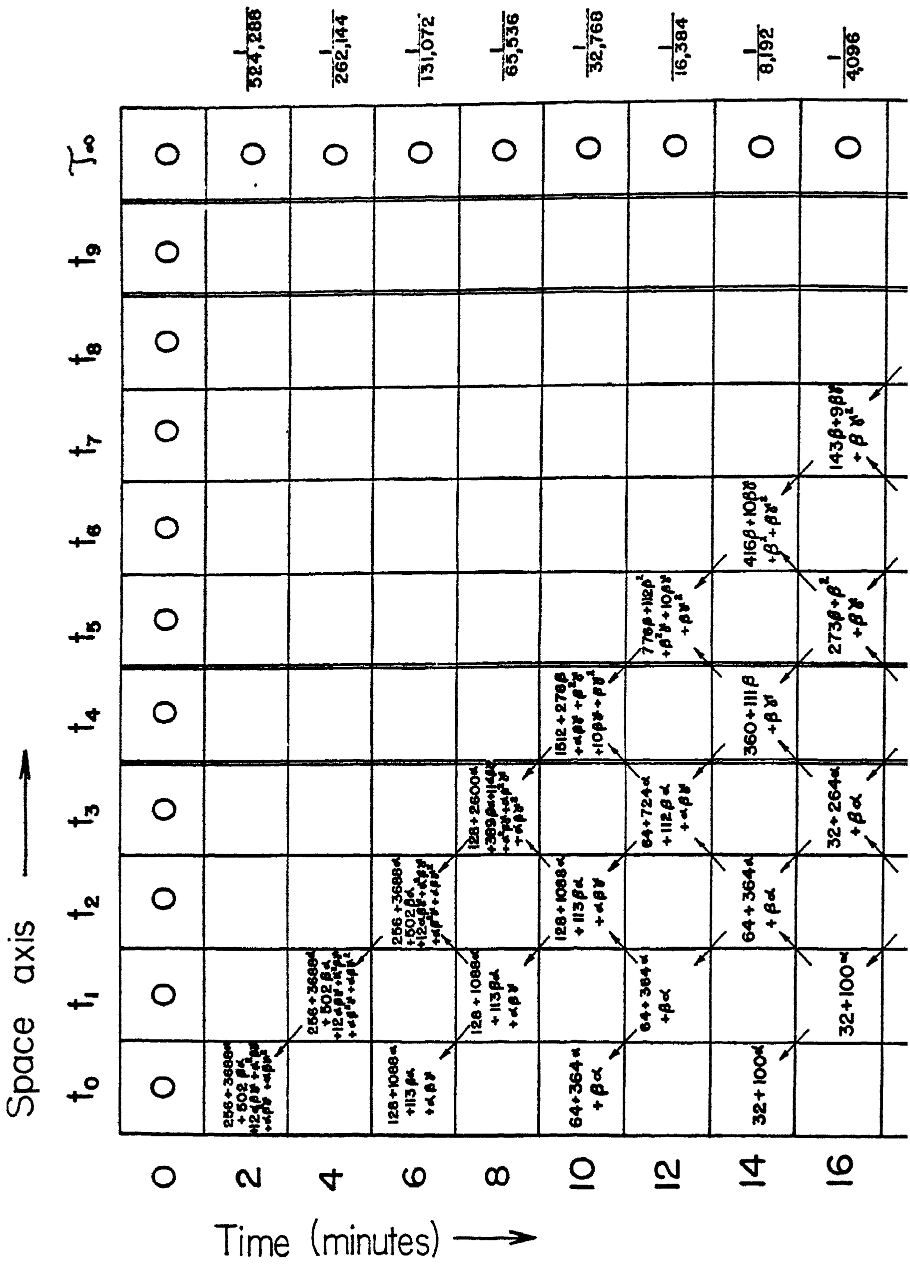
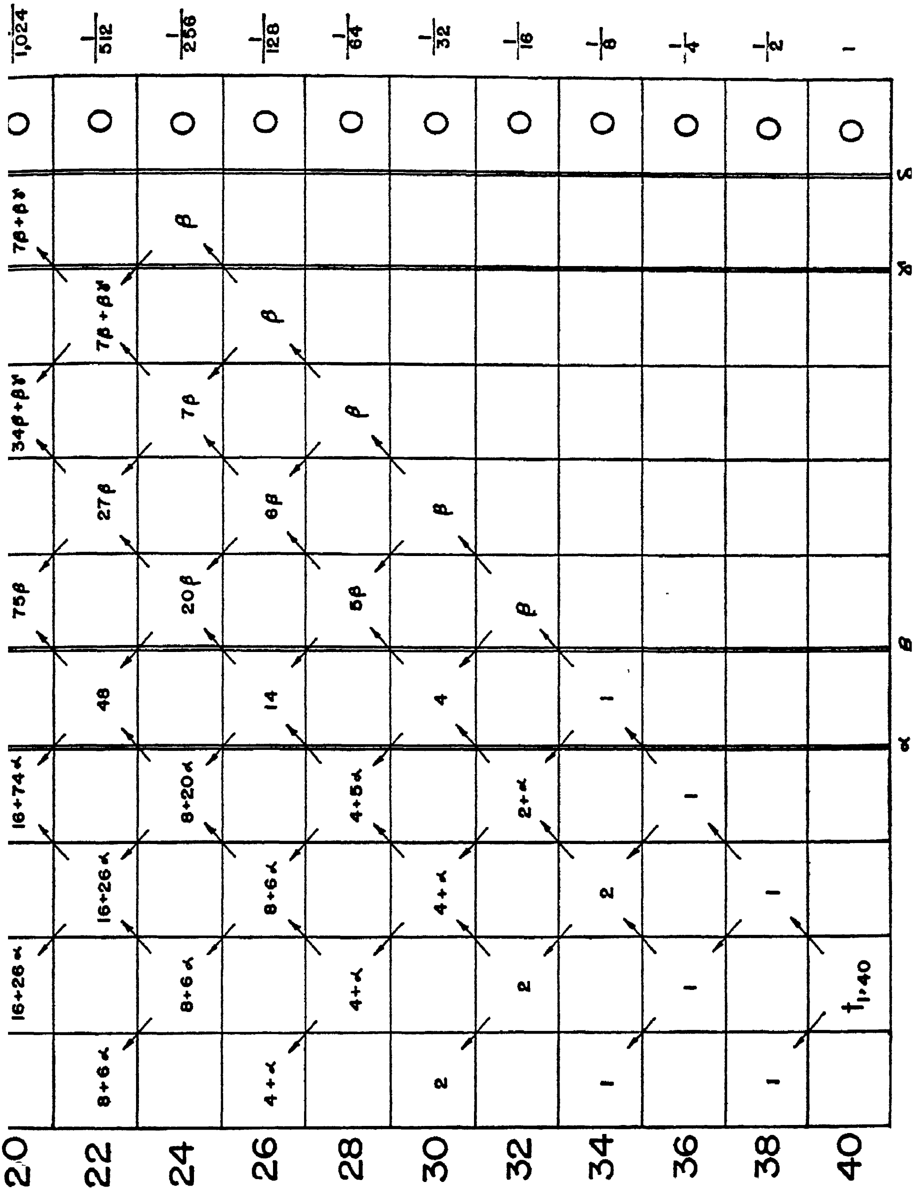


FIG. 6–6. MATRIX OR WORK SHEET

The completed matrix for the solution of the refrigerator wall problem. Because 50° F hand column and the top row are all zeros. The 50° F will have to be added to the solution



FOR PROBLEM OF FIG. 6-5.

has been subtracted from all initial and boundary temperatures the entries in the right when it is obtained.

$$t_{x,\theta+\Delta\theta} = \frac{t_{x-1,\theta} + t_{x+1,\theta}}{2} \qquad (x = 1, 2, 3, 5, 6, 7, 8)$$

$$t_{4,\theta+\Delta\theta} = \frac{\Delta x_w}{\Delta x_w + \Delta x_c\left(\frac{k_w}{k_c}\right)} t_{3,\theta} + \frac{\Delta x_c}{\Delta x_c + \Delta x_w\left(\frac{k_c}{k_w}\right)} t_{5,\theta} \qquad \text{(Ref. 3)}$$

$$t_{9,\theta+\Delta\theta} = \frac{\left(\frac{k_w}{h_c}\right)}{\Delta x_w + \left(\frac{k_w}{h_c}\right)} t_{8,\theta} + \frac{\Delta x_w}{\Delta x_w + \left(\frac{k_w}{h_c}\right)} t_{\infty,\theta}$$

or

$$t_{4,\theta+\Delta\theta} = \frac{\alpha}{2} t_{3,\theta} + \frac{\beta}{2} t_{5,\theta} \qquad \alpha = 0.316, \quad \beta = 1.683$$

$$t_{9,\theta+\Delta\theta} = \frac{\epsilon}{2} t_{8,\theta} + \frac{\delta}{2} \tau_{\infty,\theta} \qquad \epsilon = 0.460, \quad \delta = 1.54$$

Because the boundary and initial temperatures enter linearly into the solution, it is expedient to subtract 50 from all temperatures and then add 50 to the solution. With this change the finished matrix appears as shown in Fig. 6–6.

The Matrix. Equations for the temperatures can be read from the completed matrix. Figure 6–6 shows the matrix for 42 minutes. However, the temperatures at lesser times can be read by shifting the time origin downward. For example, from this matrix the following equations are obtained:

$$t_{1,6} = \frac{1}{2} t_{0,4} + \frac{1}{8} t_{0,0}$$

$$t_{1,10} = \frac{1}{2} t_{0,8} + \frac{1}{8} t_{0,4} + \frac{2}{32} t_{0,0}$$

$$t_{1,16} = \frac{1}{2} t_{0,14} + \frac{1}{8} t_{0,10} + \frac{2}{32} t_{0,6} + \frac{(4+\alpha)}{128} t_{0,2}$$

$$t_{1,22} = \frac{1}{2} t_{0,20} + \frac{1}{8} t_{0,16} + \frac{2}{32} t_{0,12} + \frac{(4+\alpha)}{128} t_{0,8} + \frac{8+6\alpha}{512} t_{0,4} + \frac{(16+26\alpha)}{2048} t_{0,0}$$

Figure 6–7 shows a graph of t_1 and q/A as functions of time as obtained by this technique.

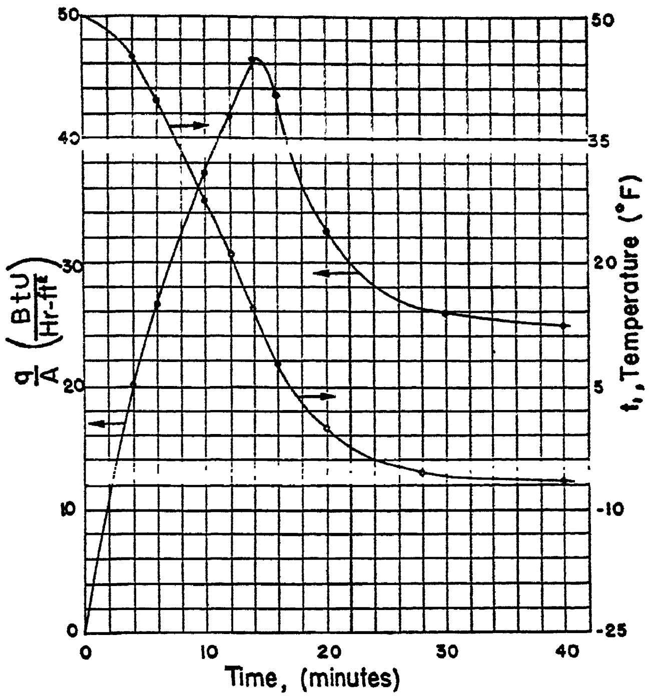


FIG. 6–7. RESULTS OF CALCULATIONS BY THE MATRIX OF FIG. 6–6.

The temperature and heat flow in the refrigerator wall are computed from the matrix shown in Fig. 6–6.

ACCURACY OF THE NUMERICAL PROCESS

Possible Improvements. The construction of the work sheet showing the zones of influence reveals the interesting fact that in the numerical averaging procedure adopted, the temperature obtained, say, at $t_{e,8}$ in Fig. 6–4, does not appear to depend upon the initial values of $t_{c,1}$, $t_{e,1}$, $t_{g,1}$, or $t_{i,1}$. However, the temperature at an instant later does depend upon these values. Thus it is to be expected that unless the initial temperature distribution is very smooth, the computed value of $t_{e,8}$ will oscillate until the boundary temperatures exert the predominant influence. *It would appear that a better averaging technique should be adopted.* This procedure was proposed in reference 5 and it was suggested that the time and space increments be related by $M = (\Delta x)^2/a\,\Delta\theta$ where M could be any value greater than or equal to 2. Of course, when M is assigned a value greater than 2 the time increment becomes smaller for a given value of Δx. That is, if Δx remains fixed, a larger number of calculations will be

required to advance a given total interval in time. For example, if $M = 3$, we will find that 1.5 times as many calculations must be made as for $M = 2$. On the other hand, since the value of the computed temperature at a point should no longer oscillate, perhaps larger increments Δx will be permitted for a given accuracy.

Study of Errors. In order to investigate this matter the derivations of the working equations were examined in a fashion similar to that of reference 7.

Define certain errors, E_1, E_2, E_3 and E_θ, as follows:

$$E_1 = \left(\frac{\partial t}{\partial x}\right)_{x-\frac{\Delta x}{2},\theta} - \frac{t_{x,\theta} - t_{x-\Delta x,\theta}}{\Delta x} \tag{6–12}$$

$$E_2 = \left(\frac{\partial t}{\partial x}\right)_{x+\frac{\Delta x}{2},\theta} - \frac{t_{x+\Delta x,\theta} - t_{x,\theta}}{\Delta x} \tag{6–13}$$

$$E_3 = \left(\frac{\partial^2 t}{\partial x^2}\right)_{x,\theta} - \frac{\left(\frac{\partial t}{\partial \theta}\right)_{x+\frac{\Delta x}{2},\theta} - \left(\frac{\partial t}{\partial x}\right)_{x-\frac{\Delta x}{2},\theta}}{\Delta x} \tag{6–14}$$

$$E_\theta = \left(\frac{\partial t}{\partial \theta}\right)_{x,\theta} - \frac{t_{x,\theta+\Delta\theta} - t_{x,\theta}}{\Delta\theta} \tag{6–15}$$

The E's represent the error incurred in replacing a derivative by a ratio of finite differences. If these equations (6–12) to (6–15) are substituted into equation (6–1) there results:

$$t_{x,\theta+\Delta\theta} = \frac{t_{x+\Delta x,\theta} + (M-2)t_{x,\theta} + t_{x-\Delta x,\theta}}{M} + \frac{1}{M}\,[E_3(\Delta x)^2 + (E_2 - E_1)\,\Delta x - ME_\theta\,\Delta\theta] \tag{6–16}$$

Now the errors defined in equations (6–12) to (6–15) depend only on one variable at a time, that is, E_1 is a function of x or of θ only, etc.; hence we may consider the expansion of the function $t(x,\theta)$ in a Taylor series first about x with θ constant and then about θ with x constant is accomplished.

$$t_\theta(x + \Delta x) = t_\theta(x) + \Delta x t'_\theta(x) + \Delta x^2 \frac{t''_\theta(x)}{2!} + \cdots \tag{6–17}$$

$$t_\theta(x - \Delta x) = t_\theta(x) - \Delta x t'_\theta(x) + (\Delta x)^2 \frac{t''_\theta(x)}{2!} + \cdots \tag{6–18}$$

$$t'_\theta\left(x + \frac{\Delta x}{2}\right) = t'_\theta(x) + \frac{\Delta x}{2}\, t''_\theta(x) + \frac{(\Delta x)^2}{4}\frac{t'''(x)}{2!} + \cdots \tag{6–19}$$

$$t'_\theta\left(x - \frac{\Delta x}{2}\right) = t'_\theta(x) - \frac{\Delta x}{2} t''_\theta(x) + \frac{(\Delta x)^2}{4} \frac{t'''(x)}{2!} + \cdots \qquad (6\text{–}20)$$

$$t_x(\theta + \Delta\theta) = t_x(\theta) + \Delta\theta t'_x(\theta) + (\Delta\theta)^2 \frac{t''(\theta)}{2!} + \cdots \qquad (6\text{–}21)$$

From these equations and the definitions of the errors, the sum of the errors can now be written. Then writing E' as the total error, that is, the last term in equation (6–16),

$$E' = \frac{1}{M} [E_3(\Delta x)^2 + (E_2 - E_1)\,\Delta x - ME_\theta\,\Delta\theta] \qquad (6\text{–}22)$$

Limit on Error. When equations (6–17) to (6–21) are substituted into equations (6–12) to (6–15) and these in turn are substituted into equation (6–22), there result two infinite series whose leading terms contain $(\partial^2 t/\partial\theta^2)$ and $(\partial^4 t/\partial x^4)$. *By the remainder theorem, the series are less in absolute value than the leading terms evaluated at their maximum values.* Therefore,

$$|E'| \leq \left| \frac{(\Delta\theta)^2}{2} \left[\left(\frac{\partial^2 t}{\partial\theta^2}\right) - \frac{a^2 M}{6} \left(\frac{\partial^4 t}{\partial x^4}\right) \right] \right| \qquad (6\text{–}23)$$

Equation (6–23) yields the maximum value of the error incurred at any one step in the numerical process. These errors accumulate in a very complex fashion, but in general, if one is making the computations for a period of time, say $n\,\Delta\theta$, *then the absolute value of the error will not exceed the total of the absolute value of the errors at each step.*

$$\left|\sum E\right| \leq \left|\sum_1^h E'_m\right| \qquad (6\text{–}24)$$

If the largest value of E' is denoted by E'', then

$$E \leq nE'' \qquad (6\text{–}25)$$

The error of one step, multiplied by the number of steps should be held to a minimum. Therefore:

$$E \leq nE'' = (n\,\Delta\theta) \frac{\Delta\theta}{2} \left[\frac{\partial^2 t}{\partial\theta^2} - \frac{a^2 M}{6} \frac{\partial^4 t}{\partial x^4} \right] \qquad (6\text{–}26)$$

The maximum total error in any one solution calculated for a specified time interval $\theta = n\,\Delta\theta$ is proportional to the time increment:

$$E = \theta \cdot \frac{\Delta\theta}{2} \left[\frac{\partial^2 t}{\partial\theta^2} - \frac{a^2 M}{6} \frac{\partial^4 t}{\partial x^4} \right] \qquad (6\text{–}27)$$

The error due to replacing the partial derivative with respect to time by a difference quotient is independent of M. The error due to re-

placing the second derivative with respect to space increases with increasing M.

Number of Numerical Operations. The total number of arithmetical operations is equal to the number of steps, n, multiplied by the number of operations at each step. The operations for each step for $M = 2$ are:

(1) At an interior point of a homogeneous solid:
   (a) One addition (b) One multiplication or division
(2) At a discontinuity:
   (a) Two multiplications (b) One addition

Thus, if $M = 2$, one performs $2n$ or $3n$ operations for each Δx and if the total dimension is x, one performs p operations, where:

$$\frac{2nx}{\Delta x} \leq p \leq \frac{3nx}{\Delta x} \tag{6-28}$$

If $M \neq 2$, one is required to perform at all points three multiplications and one addition. The total number of operations is therefore given by $p = 4nx/\Delta x$

Summarizing:

For $M = 2$

$$\frac{2nx}{\sqrt{2a}\,\sqrt{\Delta\theta}} \leq p \leq \frac{3nx}{\sqrt{2a}\,\sqrt{\Delta\theta}} \tag{6-29}$$

For $M \neq 2$

$$p = \frac{4nx}{\sqrt{Ma}\,\sqrt{\Delta\theta}} \tag{6-30}$$

Therefore, if a time increment $\Delta\theta$ is chosen the same for both cases, the ratio of operations required for each of the two cases, $M = 2$ and $M \neq 2$,

$$\frac{1}{2}\sqrt{\frac{M}{2}} \leq \frac{P_{M=2}}{P_{M\neq 2}} \leq \frac{3}{4}\sqrt{\frac{M}{2}} \tag{6-31}$$

When the ratio $(P_{M=2}/P_{M\neq 2})$ is greater than unity, it means that a larger number of operations is to be performed when $M = 2$ than when $M \neq 2$. It is seen that if M is made very large this ratio will exceed unity. If every lamination represents a discontinuity of properties, the ratio of operations will equal unity when $M = 3\frac{5}{9}$. For a homogeneous solid the number of operations will be equal at $M = 8$. Therefore, under no circumstances will there be any gain from taking M greater than 2 but less than $3\frac{5}{9}$.

If the machine upon which these computations are to be performed has peculiar characteristics, as for example, requiring a much longer

time to multiply than to add (as is common in most computing machines), a different value of M will be found as the optimum. On the other hand, referring again to equation (6–27), if the curve of temperature versus distance is less curved than a quartic, i.e., the fourth derivative vanishes, no error is incurred in taking M or Δx very large.

Conclusion. If it is desired to know the temperatures in a solid as a function of the initial and surface temperatures, the proposed short cut will be useful. It will be of greatest utility when it is desired to ascertain the effect of differing boundary and initial temperatures upon the temperature at a particular interior position.

The error in the numerical integration method is proportional to the time increment $\Delta\theta$.

The choice of the reciprocal of the Fourier difference modulus $M = (\Delta x)^2/a\,\Delta\theta$ other than 2 requires more labor for a given accuracy than for $M = 2$ unless $M > 3\frac{5}{9}$ and sometimes unless $M > 8$.

The error in the numerical method depends upon the magnitudes of $\dfrac{\partial^2 t}{\partial \theta^2}$ and on $a^2 M \dfrac{\partial^4 t}{\partial x^4}$. If the temperature variations with space are less curved than the graph of a quartic equation, the value of M may be chosen as large as desired without error. Under such circumstances M should be chosen at least equal to 8.

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CHAPTER 7

TWO PROBLEMS IN BUILDING HEATING SOLVED NUMERICALLY

G. M. DUSINBERRE *

Synopsis. One transient and one steady state problem, solved otherwise in the literature, have been selected to demonstrate practical numerical methods in this field and to permit comparison as to facility and accuracy. The transient problem deals with the rate of cooling of a building when the heating plant is shut down, and the rate of heating up when the plant is operated at maximum capacity. The steady state problem deals with heat supply and comfort conditions in a room when some account is taken of the effect of radiant temperatures. The methods developed are not limited to the present type problems, but have general applicability.

Introduction. In applying numerical analysis to heat transfer problems, the first step is to subdivide the physical system into a number of regions, to each of which a representative temperature may be assigned. This process calls for some judgment. If the subdivision is too crude, the accuracy may be low. Or if the accuracy is satisfactory under crude assumptions, we might have saved time by the use of calculus. On the other hand, if the number of subdivisions is unduly large, we may have wasted time in obtaining a degree of accuracy not warranted by the data or by the practical application of the solution.

When dealing with a homogeneous system, the subdivision is made on a geometrical basis, into squares, cubes, triangles, etc., and the problems are classified as one-, two-, or three-dimensional. The examples in this paper have the advantage of showing a more general scheme of subdivision, on a *physical rather than a geometrical basis, and with any number of effective dimensions.***

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** Like moment distribution (Chapter 1) and grid analysis presented in Chapter 2 the numerical procedure given here maintains a close association with the physical body. *Ed.*

Basic Equations

Transmittance. Having subdivided our system, we calculate an over-all transmittance, K, between each pair of regions which can exchange heat. The term *transmittance* is understood to include the effect of whatever modes of heat transfer are appropriate. In the present problems we take these quantities as constant.* Then, for example, the heat flow from region 2 to region 1 is given by:

$$Q'_{21} = K_{21}(T_2 - T_1) \tag{7-1}$$

where

$$Q' = \text{heat flow, Btu/hr}$$
$$K = \text{transmittance, Btu/hr °F}$$
$$T = \text{temperature, °F}$$

The subscript *21* is to be read "two, one".

There may be a heat supply Q'_s within a region, independent of the temperature. For region 1 we designate this Q'_{s1}, which is in Btu/hr. Then the total heat flow to region 1, for example, is:

$$K_{21}(T_2 - T_1) + \cdots\cdots + K_{n1}(T_n - T_1) + Q'_{s1} = Q'_1 \tag{7-2}$$

Steady Flow. If the heat flow is steady in time, Q'_1 is zero. If the heat flow is transient, but the heat capacity of the region is relatively very small, so that it is substantially in equilibrium with its surroundings at all times, Q'_1 may be negligible. For these cases, rearranging equation (7–2):

$$K_{21}T_2 + \cdots\cdots + K_{n1}T_n + Q'_{s1} - \Sigma K_1 T_1 = 0 \tag{7-3}$$

$$T_1 = \frac{K_{21}}{\Sigma K_1} T_2 + \cdots\cdots + \frac{K_{n1}}{\Sigma K_1} T_n + \frac{Q'_{s1}}{\Sigma K_1} \tag{7-4}$$

$$T_1 = F_{21}T_2 + \cdots\cdots + F_{n1}T_n + Q_1 \tag{7-5}$$

in which the value of Q and of each F is evident from a comparison of equations (7–4) and (7–5).

Transient Flow. But if we have a transient problem in which heat capacity is important, we have to take account of it. The heat capacity, C, (Btu/° F) is calculated for each region. Then if we can assume that the heat flow will be nearly constant over a small time interval, and again taking region 1 for example we obtain:

$$Q'_1 \Delta\theta = C_1(T'_1 - T_1) \tag{7-6}$$

* In a paper by the author in Trans. ASME (Nov. 1945), p. 703–12 methods are outlined which can be used in some cases where the physical properties vary significantly with temperature.

in which $\Delta\theta$ denotes the time interval, T is the temperature at the beginning of that interval and T' is the temperature at the end of the interval. Substituting in equation (7–2) and rearranging terms, we have:

$$T_1' = \frac{K_{21}\,\Delta\theta}{C_1}T_2 + \cdots\cdots + \frac{K_{n1}\,\Delta\theta}{C_1}T_n + \frac{Q_{s1}'\,\Delta\theta}{C_1} + \left[1 - \frac{\Sigma K_1\,\Delta\theta}{C_1}\right]T_1 \tag{7–7}$$

$$T_1' = F_{21}T_2 + \cdots\cdots + F_{n1}T_n + Q_1 + F_{11}T_1 \tag{7–8}$$

Here the Q *and the* F *terms have different meanings from those in equation (7–5).* Their definitions are obtained by comparing equations (7–7) and (7–8).

Criterion for Convergence. We must give special attention to the term F_{11}. All the other F terms are inherently positive, but it is possible to choose $\Delta\theta$ so as to make F_{11} negative. This would be as much as to say: the hotter region 1 is at the beginning of the interval $\Delta\theta$, the colder it will be at the end. Practically, this will give rise to an erroneous oscillation or divergence in the calculated values. (The same sort of difficulty can arise with infinite series.) So we should take as a criterion:

$$F_{11} \geqq 0 \tag{7–9}$$

whence:

$$\Delta\theta \leqq \frac{C_1}{\Sigma K_1} \tag{7–10}$$

An equation similar to (7–10) is written for each region, and $\Delta\theta$ must be chosen to satisfy all these. If a cyclic operation is under study, $\Delta\theta$ should be made some fraction of the cycle.

Step-by-Step. On writing an equation similar to (7–5) or (7–8) for each region, we are prepared to carry out a step-by-step analysis of any prescribed transient situation. The procedure will be illustrated by examples.

Problem I

Statement of Problem. We have a small industrial building, 50 ft square and 50 ft high above ground, of prescribed construction equipped with a heating plant of prescribed capacity. Between 9 : 00 A.M. and 4 : 30 P.M. the interior temperature is to be maintained at 70° F; at other times it may vary. Ventilating air is to be provided at all times, at the rate of 4000 cfm.

It is desired to investigate the rate at which the building will cool off and heat up under conditions (1) and (2), as follows:

(1) Outdoor temperature is constant at 30° F. Heating plant is shut down at 4 : 30 P.M. and is started at full capacity in time to have the interior at 70° by 9 : 00 A.M. It is required to find the time when the plant should be started.*

(2) Beginning at 4 : 30 P.M., the outdoor temperature falls at 2°/hr to 15° F, where it remains constant. Other data and requirements are the same as in part (1).

Assumptions. The following assumptions are made:

(*a*) Conduction through the ground is negligible.

(*b*) Conduction from floors to outside walls and from partitions to roof is negligible.

(*c*) Heat capacity of windows and of inside air is negligible.

We subdivide the system as follows:

o — outdoors
w — outside walls
r — roof
i — inside partitions and floors
a — inside air

Figure 7–1 shows the assumed paths of heat flow.

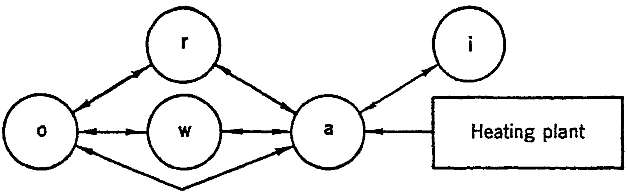

FIG. 7–1. HEATING PLANT DIAGRAM.

Transmittances. From the dimensions and materials given, and from handbook coefficients, we calculate the following quantities where the subscripts refer to transfer between the regions of Fig. 7–1:

$$K_{wo} = K_{ow} = 6840 \text{ Btu/hr } ° \text{F}$$
$$K_{wa} = K_{aw} = 4970 \text{ Btu/hr } ° \text{F}$$
$$K_{ro} = K_{or} = 7450 \text{ Btu/hr } ° \text{F}$$
$$K_{ra} = K_{ar} = 3220 \text{ Btu/hr } ° \text{F}$$

(Each of the above includes the appropriate surface coefficient plus one-half the resistance of the wall or roof itself.)

$$K_{oa} = K_{ao} = 6580 \text{ Btu/hr } ° \text{F}$$

* Up to this point, the problem is similar to that outlined by B. F. Raber and F. W. Hutchinson in their text, *Refrigeration and Air Conditioning Engineering* (Wiley), pp. 143–146. It will be of interest and value to compare the methods of solution. Part (2) of this problem cannot be handled by the method of this reference.

(This is made up of transmission through the window area plus convection to the ventilating air.)

$$K_{ai} = K_{ia} = 74{,}200 \text{ Btu/hr °F}$$

Heat Capacities

$$C_w = 169{,}300 \text{ Btu/° F}$$
$$C_r = 17{,}500 \text{ Btu/° F}$$
$$C_i = 189{,}200 \text{ Btu/° F}$$

Convergence. We now find the limits on $\Delta\theta$:

$$\text{For the walls, } \Delta\theta \leqq \frac{169{,}300}{6840 + 4970} = 14.3 \text{ hr}$$

$$\text{For the roof, } \Delta\theta \leqq \frac{17{,}500}{7450 + 3220} = 1.64 \text{ hr}$$

$$\text{For the inside walls, etc., } \Delta\theta \leqq \frac{189{,}200}{74{,}200} = 2.55 \text{ hr}$$

Then a convenient $\Delta\theta$ will be 1.5 hr, and this value also divides evenly into 24 hr.

Heat Balance Equations. As we are neglecting the effect of C_a, we use equation (7–5):

$$\Sigma K_a = 4970 + 3220 + 6580 + 74{,}200 = 88{,}970$$

$$F_{wa} = \frac{4970}{88{,}970} = 0.056$$

$$F_{ra} = \frac{3220}{88{,}970} = 0.036$$

$$F_{oa} = \frac{6580}{88{,}972} = 0.074$$

$$F_{ia} = \frac{74{,}200}{88{,}970} = 0.834$$

$$T_a = 0.056T_w + 0.036T_r + 0.074T_o + 0.834T_i + Q_a \qquad (7\text{–}11)$$

For the other subdivisions, use equation (8):

$$F_{ow} = 6840 \times \frac{1.5}{169{,}300} = 0.061$$

$$F_{aw} = 4970 \times \frac{1.5}{169{,}300} = 0.044$$

$$F_{ww} = 1 - 0.061 - 0.044 = 0.895$$

$$T'_w = 0.061T_o + 0.044T_a + 0.895T_w \tag{7-12}$$

$$F_{or} = 7450 \times \frac{1.5}{17{,}500} = 0.638$$

$$F_{ar} = 3220 \times \frac{1.5}{17{,}500} = 0.276$$

$$F_{rr} = 1 - 0.638 - 0.276 = 0.086$$

$$T'_r = 0.638T_o + 0.276T_a + 0.086T_r \tag{7-13}$$

$$F_{ai} = 74{,}200 \times \frac{1.5}{189{,}200} = 0.589$$

$$F_{ii} = 1 - 0.589 = 0.411$$

$$T'_i = 0.589T_a + 0.411T_i \tag{7-14}$$

Computation Scheme. We can now set up the F values in tabular form as a guide in computation:

	o	w	r	i	a
o	—	.061	.638	—	.074
w	—	.895	—	—	.056
r	—	—	.086	—	.036
i	—	—	—	.411	.834
a	—	.044	.276	.589	—

Each column represents one of the equations (7–11) to (7–14).

Steady-State Conditions. Before starting the calculation we need to know the steady-state temperatures of the mid-points of the outside walls and roof.

$$T_w = \frac{6840 \times 30 + 4970 \times 70}{11{,}810} = 46.8^\circ \text{ F}$$

$$T_r = \frac{7450 \times 30 + 3220 \times 70}{10{,}670} = 42.1^\circ \text{ F}$$

Then at 3 : 00 P.M. the temperatures in the system are:

θ	o	w	r	i	a
3 : 00 P.M.	30.0	46.8	42.1	70.0	70.0

Let us predict the temperatures after the interval $\Delta\theta$, that is, at 4 : 30. The first term on the right of equation (12) is $0.061T_o =$

$0.061 \times 30.0 = 1.8$, and this is a contribution toward T'_w. Show this result as follows:

θ	o	w	r	i	a
3 : 00 P.M.	30.0	46.8	42.1	70.0	70.0
		1.8			

Continuing with the remaining items in the w, r, and i columns, we would have:

θ	o	w	r	i	a
3 : 00 P.M.	30.0	46.8	42.1	70.0	70.0
		1.8	19.2		
		41.9			
			3.6		
				28.8	
		3.1	19.3	41.2	

Temperatures at 4 : 30 P.M. Adding these results, we get the temperatures at 4 : 30 P.M.:

θ	o	w	r	i	a
4 : 30 P.M.		46.8	42.1	70.0	

which merely confirms the steady state which has been assumed. T_o can of course be written in, as it does not depend on anything going on in our building. Now T_a has to be calculated from conditions at 4 : 30, not at 3 : 00 P.M. Multiplying the other temperatures by the appropriate F values we get:

a
2.2
2.6
1.5
58.4
64.7° F

The sum is the temperature which the air will attain shortly after 4 : 30 when the heating plant is shut down. To maintain the 70° steady state we have to add a $Q_a = 70.0 - 64.7 = 5.3°$ F, and from this, $Q'_{sa} = Q_a \Sigma K_a = 5.3 \times 88{,}970 = 471{,}000$ Btu/hr. This is the rate at which the heating plant must operate to maintain the assumed daytime condition.

Temperatures to 3 A.M. Now we have all the temperatures for 4 : 30. By repetition of the same series of calculations, which involve nothing but multiplication and addition, we can find the temperatures at any number of succeeding intervals of $\Delta\theta$. Results, to 3 : 00 A.M., are shown below:

θ	o	w	r	i	a
3:00	30.0	46.8	42.1	70.0	70.0
P.M.		1.8	19.2		2.2
		41.9			2.6
			3.6		1.5
				28.8	58.4
		3.1	19.3	41.2	
4:30	30.0	46.8	42.1	70.0	64.7
P.M.		1.8	19.2		2.2
		41.9			2.6
			3.6		1.5
				28.8	55.7
		2.8	17.8	38.1	
6:00	30.0	46.5	40.6	66.9	62.0
P.M.		1.8	19.2		2.2
		41.6			2.6
			3.5		1.4
				27.4	53.5
		2.7	17.1	36.5	
7:30	30.0	46.1	39.8	63.9	59.7
P.M.		1.8	19.2		2.2
		41.2			2.6
			3.4		1.4
				26.2	51.1
		2.6	16.5	35.1	
9:00 P.M.	30.0	45.6	39.1	61.3	57.3

θ	o	w	r	i	a
9:00	30.0	45.6	39.1	61.3	57.3
P.M.		1.8	19.2		2.2
		40.8			2.5
			3.4		1.4
				25.2	49.2
		2.5	15.8	33.8	
10:30	30.0	45.1	38.4	59.0	55.3
P.M.		1.8	19.2		2.2
		40.4			2.5
			3.3		1.4
				24.2	17.4
		2.4	15.3	32.6	
12:00	30.0	44.6	36.8	56.8	53.5
P.M.		1.8	19.2		2.2
		40.0			2.5
			3.3		1.3
				23.4	45.8
		2.4	14.8	31.5	
1:30	30.0	44.2	37.3	54.9	51.8
A.M.		1.8	19.2		2.2
		39.6			2.4
			3.2		1.3
				22.6	44.3
		2.3	14.3	30.5	
3:00 A.M.	30.0	43.7	36.7	53.1	50.2

Starting Heating Plant. For the next step of the problem, some trial-and-error is necessary in this as in other methods of numerical solution. We have to guess a time for starting the heating plant, and, if this does not come out right at 9 : 00 A.M., we must try another time in the direction indicated by the error, and then interpolate if necessary. The outdoor air still remains constant at 30° F. We will show here only the correct guess.

The maximum capacity of the heating plant is specified as 820,000 Btu/hr. For this condition, $Q_a = 820{,}000/88{,}970 = 9.2°$. This will be added to find each T_a value in accordance with equation (7–11). For example, if we should start the plant at 1 : 30 A.M., T_a becomes $51.8 + 9.2 = 61.0$. The calculation would then proceed as follows:

θ	o	w	r	i	a
1:30	30.0	44.2	37.3	54.9	61.0
A.M.		1.8	19.2		2.2
		39.6			2.5
			3.2		1.4
				22.6	48.9
		2.7	16.8	36.0	
					9.2
3:00	30.0	44.1	39.2	58.6	64.2
A.M.		1.8	19.2		2.2
			39.5		2.5
			3.4		1.5
				24.1	51.5
		2.8	17.7	37.8	
					9.2
4:30	30.0	44.1	40.3	61.9	66.9
A.M.		1.8	19.2		2.2
		39.5			2.5
			3.5		1.5
				25.4	54.0
		2.9	18.5	39.4	
					9.2
6:00	30.0	44.2	41.2	64.8	69.4
A.M.					

θ	o	w	r	i	a
6:00	30.0	44.2	41.2	64.8	69.4
A.M.		1.8	19.2		2.2
		39.6			2.5
			3.5		1.5
				26.6	56.1
		3.0	19.1	40.7	
					9.2
7:30	30.0	44.4	41.8	67.3	71.5
A.M.		1.8	19.2		2.2
		39.7			2.5
			3.6		1.5
				27.6	58.1
		3.1	19.7	42.1	
					9.2
9:00	30.0	44.6	42.5	69.7	73.5
A.M.					

The desired condition has been substantially attained. At 9 : 00 A.M. we can begin to operate the plant at a lower rate, reducing T_a. For the significance of the column headings see Fig. 7–1, p. 107.

Calculus Solution. An alternative solution giving the same result can be found in the Raber and Hutchinson reference. A third solution can be found by simplifying the problem and applying elementary calculus. The total transmittance of the building, for walls, roof, windows, and ventilation, can be found to be $K_t = 11{,}710$ Btu/hr °F. We neglect all heat capacities except the interior. The rule for Newtonian cooling of an object to which a single temperature can be assigned, in a constant ambient temperature (T_0), is

$$(T_2 - T_0) = (T_1 - T_0) \exp \frac{-K_t}{C_i} (\theta_2 - \theta_1) \tag{7–15}$$

Substituting $T_0 = 30$, $T_1 = 70$, $K_t = 11{,}710$, $C_i = 189{,}200$, $\theta_1 =$ 4 : 30 P.M., $\theta_2 =$ 1 : 30 A.M., we get $T_2 - T_0 = 22.8$, or $T_2 = 52.8$, which may be compared with the numerical results of 54.9 for the interior walls and 51.8 for the interior air.

The corresponding rule for heating is readily derived:

$$(\theta_3 - \theta_2) = \frac{C_i}{K_t} \ln \frac{Q' - K_t(T_2 - T_0)}{Q' - K_t(T_3 - T_0)} \tag{7–16}$$

When we let $T_2 - T_0 = 22.8$, as found above, $T_3 = 70$ and $Q' =$ 820,000, we find $\theta_3 - \theta_2 = 7.44$ hr, and $\theta_3 =$ 8 : 56 A.M.

Conclusions from Problem I. It appears, then, that our analysis is more refined than necessary for a reasonable solution of part (1) of our problem, and has merely served its purpose of illustrating a method. But now suppose that the problem requires a finer subdivision. Suppose, for example, that we needed to know the surface temperatures inside the exterior walls and roof, for a study of sweating conditions. *The method would be to split the walls and roof into layers and use the same technique with a more extensive computation form.* This procedure will not be illustrated here, but it should be evident that the only cost is a little more computation time. It should also be evident that a calculus solution would be almost intolerably complex, even if possible at all.

Again, suppose the ambient temperature varies in some arbitrary manner, as in part (2) of our problem. The calculus solution will become extremely complex, but the numerical analysis can be extended without any change in the computation form. For the data given we will find that the heating plant will have to be started a little before midnight. The calculations for starting the plant at midnight are shown in Table 7–1. Starting at 4 : 30 P.M. the temperature of the outside air drops from 30° F to 15° F at 2° per hour.

Problem II

In starting Problem I we found two steady-state temperatures very readily, because the temperatures of the adjacent regions were known. Sometimes it is necessary to find a large number of steady-state temperatures when we know the temperatures or heat inputs for only a relatively few regions. For such problems we recommend and exhibit here the "relaxation" method.

Relaxation vs Iteration. The method outlined here for transients, and also some of the numerical methods for steady states (Shortley & Weller, for example) may be characterized as iterative. The calculation, once started, proceeds in a purely mechanical way. Judgment no longer enters; "nature takes its course." This is an advantage when it is possible to make use of automatic calculating devices. *But the "relaxation" procedure is not of this sort.* The computer has some discretion and, while a correct solution will be reached in any case, the better his judgment, the sooner he will reach it.

For systems subdivided geometrically, Emmons * has treated

* H. W. Emmons, "The Numerical Solution of Heat Conduction Problems," Trans. ASME, 65, 607 (1943).

TABLE 7-1. PROBLEM I, PART (2). CALCULATION OF COOLING AND HEATING WITH VARYING OUTDOOR TEMPERATURE

θ	o	w	r	i	a
4:30	30.0	46.8	42.1	70.0	64.7
P.M.		1.8	19.2		2.0
		41.9			2.6
			3.6		1.5
				28.8	55.7
		2.8	17.8	38.1	
					0
6:00	27.0	46.5	40.6	66.9	61.8
P.M.		1.6	17.2		1.8
		41.6			2.6
			3.5		1.4
				27.4	53.1
		2.7	17.1	36.4	
					0
7:30	24.0	45.9	37.8	63.8	58.9
P.M.		1.5	15.3		1.6
		41.1			2.5
			3.2		1.3
				26.2	50.8
		2.6	16.3	34.7	
					0
9:00	21.0	45.2	34.8	60.9	56.2
P.M.		1.3	13.4		1.3
		40.4			2.5
			3.0		1.1
				25.0	48.5
		2.5	15.5	33.1	
					0
10:30	18.0	44.2	31.9	58.1	53.4
P.M.		1.1	11.5		1.1
		39.6			2.4
			2.7		1.0
				23.8	46.0
		2.3	14.7	31.4	
					9.2
12:00	15.0	43.0	28.9	55.2	59.7
A.M.		0.9	9.6		1.1
		38.5			2.4
			2.5		1.0
				22.7	48.3
		2.6	16.5	35.2	
					9.2
1:30	15.0	42.0	28.6	57.9	62.0
A.M.					

θ	o	w	r	i	a
1:30	15.0	42.0	28.6	57.9	62.0
A.M.		0.9	9.6		1.1
		37.6			2.3
			2.5		1.1
				23.8	50.4
		2.8	17.1	36.6	
					9.2
3:00	15.0	41.3	29.2	60.4	64.1
A.M.		0.9	9.6		1.1
		37.0			2.3
			2.5		1.1
				24.8	52.2
		2.8	17.7	37.8	
					9.2
4:30	15.0	40.7	29.8	62.6	65.9
A.M.		0.9	9.6		1.1
		36.4			2.3
			2.6		1.1
				25.8	53.9
		2.9	18.2	38.8	
					9.2
6:00	15.0	40.2	30.4	64.6	67.6
A.M.		0.9	9.6		1.1
		36.0			2.2
			2.6		1.1
				26.6	55.4
		3.0	18.7	39.8	
					9.2
7:30	15.0	39.9	30.9	66.4	69.0
A.M.		0.9	9.6		1.1
		35.8			2.2
			2.7		1.1
				27.3	56.7
		3.0	19.1	40.6	
					9.2
9:00 *	15.0	39.7	31.4	67.9	70.3
A.M.					

* The final line shows that the inside air temperature would reach 70.3° at 9:00 A.M. Thus, starting the heating plant at midnight would achieve the result desired. For the significance of the table headings see Fig. 7-1, p. 107.

TABLE 7-1 *Continued.* COMPUTATION SCHEME

θ	o	w	r	i	a
θ	T_o	T_w	T_r	T_i	T_a
		$T_o \times F_{ow}$	$T_o \times F_{or}$		$T'_o \times F_{oa}$
		$T_w \times F_{ww}$			$T'_w \times F_{wa}$
			$T_r \times F_{rr}$		$T'_r \times F_{ra}$
				$T_i \times F_{ii}$	$T'_i \times F_{ia}$
		$T_a \times F_{aw}$	$T_a \times F_{ar}$	$T_a \times F_{ai}$	
					Q_a
$\theta + \Delta\theta$	T'_o	T'_w	T'_r	T'_i	T'_a

rectangular coordinates and Southwell has considered triangular and hexagonal networks. For a system divided on a physical basis, Mackey * has used the work of Wright ** in solving problems in room heating by radiant panels or by convection. A problem of this type will be used for illustration. This problem is based on substantially the same data as one studied by B. F. Raber and F. W. Hutchinson in *Panel Heating and Cooling Analysis* (Wiley, 1947), p. 156.

Statement of the Problem. A room of specified dimensions and construction is to be heated by convection. The system will be subdivided as follows:

o — outdoor air
w — outside walls, with heat loss
g — windows, with heat loss
f — floor, with heat loss
i — interior walls, no heat loss
c — ceiling, no heat loss
a — inside air, heat loss by infiltration

For all boundaries of the room, the temperatures will refer to the inside surface, as this determines the radiant heat exchange. The design outdoor air temperature is 0° F.

It will be assumed that a "comfort temperature," T_s, can be determined by the following equation:

$$T_s = 0.035T_g + 0.090T_w + 0.125T_f + 0.125T_i + 0.125T_c + 0.500T_a$$

and this temperature is to be 70° F. As we are concerned here with method, we will not argue the question whether some better formulation of comfort conditions could be written.

* C. O. Mackey, *Radiant Heating and Cooling*, Part II, Cornell University Eng. Exp. Sta. Bulletin 33 (1944).

** L. T. Wright, Jr., *The Solution of Simultaneous Linear Equations by an Approximation Method*, Cornell University Eng. Exp. Sta. Bulletin 31 (1943).

Start with Arbitrary T Values. From a study of the dimensions, type of construction, angle factors, radiant and convective coefficients we can estimate all the appropriate K values. Now consider equation (7–2). There will be a Q'_s for the air and a negative Q'_s for the outdoors, and all other values of Q will be zero in the steady state. If we can write the correct values for all the T values, these conditions will be satisfied. But we have given only T_o and T_s. The problem is to find all the T values and the necessary heat supply. We start the solution by writing an arbitrary set of T values, which in general will be wrong, and by calculating the corresponding Q values, which in general will not be zero.

Now in equation (7–2), note that if we give T_n a variation, ΔT_n, making no other change, it will cause a variation in Q'_1

$$\Delta Q'_1 = K_{n1}\,\Delta T_n \qquad (7\text{–}17)$$

It will also cause a corresponding change in the Q' for every region adjacent to region n. And in region n itself, the change is:

$$\Delta Q'_n = -\Sigma K_n\,\Delta T_n \qquad (7\text{–}18)$$

These equations give a means of adjusting the temperatures, one at a time, until the Q values are zero, or as near zero as desired, whereupon the temperatures will be correct.

Computation Scheme. The procedure can best be explained by reference to the work sheet, Table 7–2. In the column for Q_o, line 2, the number 350 is the calculated value, K_{go}. In line 1, the number -617 is $-\Sigma K_o$. These, with the other items to line 7 represent the K values in the equation

$$K_{go}T_g + K_{wo}T_w + K_{fo}T_f + K_{ao}T_a - \Sigma K_oT_o + Q'_{so} = Q'_o \qquad (7\text{–}19)$$

The items in the other Q columns, lines 1–7, are obtained in the same way. The T_s column represents the "comfort equation." We can do without the Q_o column, as T_o is fixed. However, we show it here as a check on the work, since the heat received by the outside air must equal the heat supplied to the inside.

A convenient way to start this solution is to make all T values 70°. Then the system is in equilibrium and all Q values are zero. This is shown by the entries in the various columns on line 8. Next we bring T_o to its actual value. This requires a ΔT_o of -70, which is written in column T_o, line 9. Now the Q values must be changed in all regions affected by region o. We go up the T_o column to the X which guides

TABLE 7-2. COMPUTATION SCHEME AND WORK SHEET FOR PROBLEM II

	Q_o	T_o	Q_g	T_g	Q_w	T_w	Q_f	T_f	Q_i	T_i	Q_c	T_c	Q_a	T_a	T_s
(1) o	−617	X	350		83		113						71		
(2) g	350		−528	X	8		31		26		33		80		0.035
(3) w	83		8		−542	X	95		72		84		200		0.090
(4) f	113		31		95		−686	X	121		166		160		0.125
(5) i			26		72		121		−620	X	121		280		0.125
(6) c			33		84		166		121		−844	X	440		0.125
(7) a	71		80		200		160		280		440		−1231	X	0.500
(8)	0	70	0	70	0	70	0	70	0	70	0	70	0	70	70.000
(9)	43190	−70	−21500		−5810		−7910						−4970		
(10)	43190	0	−24500		−5810								−4970		
(11)	−14000		21120	−40	−320		−1240		−1040		−1320		−3200		−1.400
(12)	29190		−3380	30	−6130		−9150		−1040		−1320		−8170		68.600
(13)	355		400		1000		800		1400		2200		−6155	5	2.500
(14)	29545		−2980		−5130		−8350		360		880		−14325	75	71.100
(15)	−1130		−310		−950		6860	−10	−1210		−1660		−1600		−1.250
(16)	28415		−3290		−6080		−1490	60	−850		−780		−15925		69.850
(17)	−830		−80		5420	−10	−950		−720		−840		−2000		−0.900
(18)	27585		−3370		−660	60	−2440		−1570		−1620		−17925		68.950
(19)	142		160		400		320		560		880		−2462	2	1.000
(20)	27727		−3210		−260		−2120		−1010		−740		−20387	77	69.950
(21)	−1750		2610	−5	−40		−155		−130		−165		400		−0.175
(22)	25977		−570	25	−300		−2275		−1140		−905		−20787		69.775
(23)	−452		−121		−380		2714	−4	−484		−664		−640		−0.500
(24)	25525		−694		−680		469	56	−1624		−1569		−21427		69.275
(25)	142		160		400		320		560		880		−2462	2	1.000
(26)	25667		−531		−280		789		−1064		−689		−23889	79	70.275
(27)			−33		−81		−166		−121		844	−1	−140		−0.125
(28)			−567		−361		623		−1185		155	69	−24029		70.150
(29)			−52		−141		−242		1240	−2	−242		560		−0.250
(30)			−619		−508		381		55	68	−87		−24889		69.900
(31)	−350		528	−1	−8		−31		−26		−33		−80		−0.035
(32)	25317		−91	24	−516		350		29		−120		−24969		69.865
(33)	−83		−8		542	−1	−95		72		−84		−200		−0.090
(34)	25234		99		26	59	255		−43		−204		−25169		69.775

us to line 1. In accordance with equations (7–17) and (7–18), we multiply the value −70 by each K in line 1 and write the product in line 9. On line 10 we sum up lines 8 and 9.

Now we have to choose a point of attack in getting rid of the Q values. It is best to take the largest one first, so we try a change of −40 in T_g. This change and its effects are shown on line 11, and the summation is made on line 12. Note that this last change has affected T_s. The only way to correct this is to raise T_a, which is done on line 13. We do not attempt to make Q_a zero. The final result in that column will show how much heat has to be added to the air, which should agree with the final value of Q_o, except for sign.

Solution. Going on in this way, we get results on line 34 which can be improved, if necessary, only by working in tenths of a degree. If the present degree of precision is adequate, we can give the following answers:

Surface temperature, windows	24° F
" " outside walls	59
" " floor	56
" " interior walls	68
" " ceiling	69
Air temperature	79
"Comfort temperature," T_s	69.8
Heat required	25,200 Btu/hr

Conclusion. It will be seen that the computation scheme can be set up for any number of dimensions. Even where the subdivision is geometrical, it is often more convenient to use this form than to work over a geometrical plot of the system.

Instead of using the K values directly in the tabular form, one could use F values from equation (7–5), and each F_{nn} would be −1. This is what Wright (*loc. cit.*) has done with similar problems. The author prefers the scheme shown, because of having arrived at it independently and because it gives Q values directly. This is not a very important advantage, and Wright's setup may well be preferred.

Appendix

Temperature Distribution Through Ground

Problem Worked by J. D. Bottorf * and Y. S. Touloukian *

At Purdue University the problem of heat transfer and temperature distribution in soils in relation to buried heat sources and sinks has

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been investigated. The procedure presented above by Professor Dusinberre was adapted successfully to the solution of this problem.

Background Analysis. From the exact analytical standpoint the problem has been hardly touched. Carslaw and Jaeger * have given a solution for the ideal case of the temperature field in the area bounded internally by the horizontal pipe of radius a and otherwise unbounded, the initial temperature in the area being uniform and the pipe suddenly being raised to and maintained at a different temperature. The differential equation describing the problem and the solution as found by Carslaw and Jaeger are as follows:

$$\frac{\partial v}{\partial t} = \alpha\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \qquad (7\text{–}20)$$

$$v = 1 + \frac{2}{\pi}\int_0^\infty e^{-\alpha u^2 t}\left(\frac{J_0(ur)N_0(ua) - J_0(ua)N_0(ur)}{J_0^2(ua) + N_0^2(ua)}\right)\frac{du}{u} \qquad (7\text{–}21)$$

The evaluation of this integral involving Bessel functions of the first and second kind is extremely tedious and very time-consuming.

Practical Problem Studied. The actual problem of interest involved four horizontal tubes placed in a square grid at 2-ft centers, 8 ft below the ground surface with initial temperature boundary conditions as shown in Fig. 7–2. The differential equation describing this case is the same as that in the ideal case except for an additional variable necessitated by the specified boundary conditions.

$$\frac{\partial v}{\partial t} = \alpha\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (7\text{–}22)$$

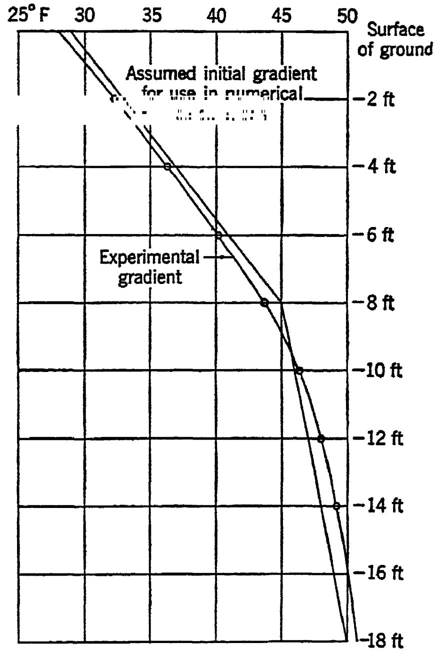

FIG. 7–2. INITIAL GROUND TEMPERATURE.

The boundary conditions here imposed by this practical problem render the equation very difficult to solve. To the best of the writers' knowledge no analytical solution of this differential equation is available which meets the boundary conditions just described.

* Phil. Mag., **26**, 473 (1938).

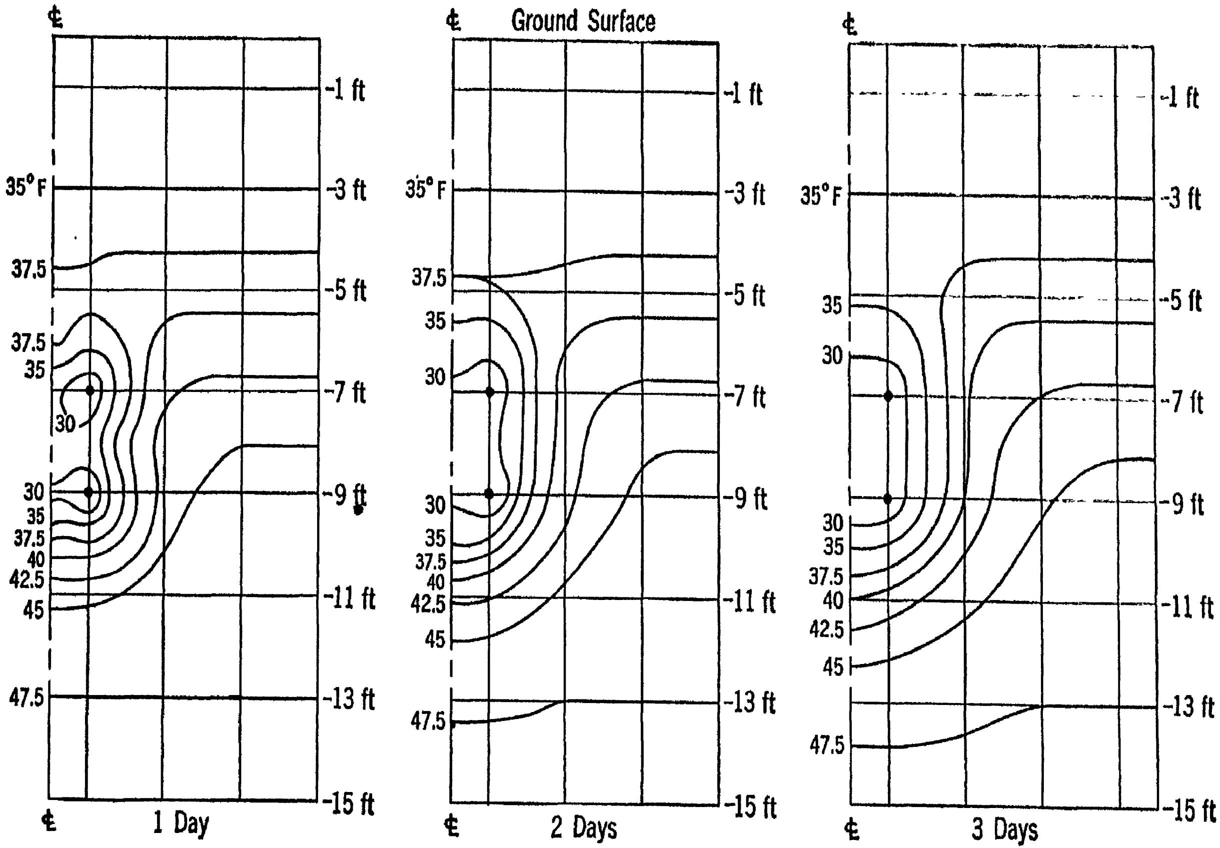

FIG. 7-3. TEMPERATURE DISTRIBUTION AFTER OPERATING PIPES AT 20° F. FOR ONE, TWO OR THREE DAYS.

Numerical Solutions. Using the procedure presented by Professor Dusinberre, temperature fields were found at six-hour intervals up to a total time of three days. The temperature gradients could just as

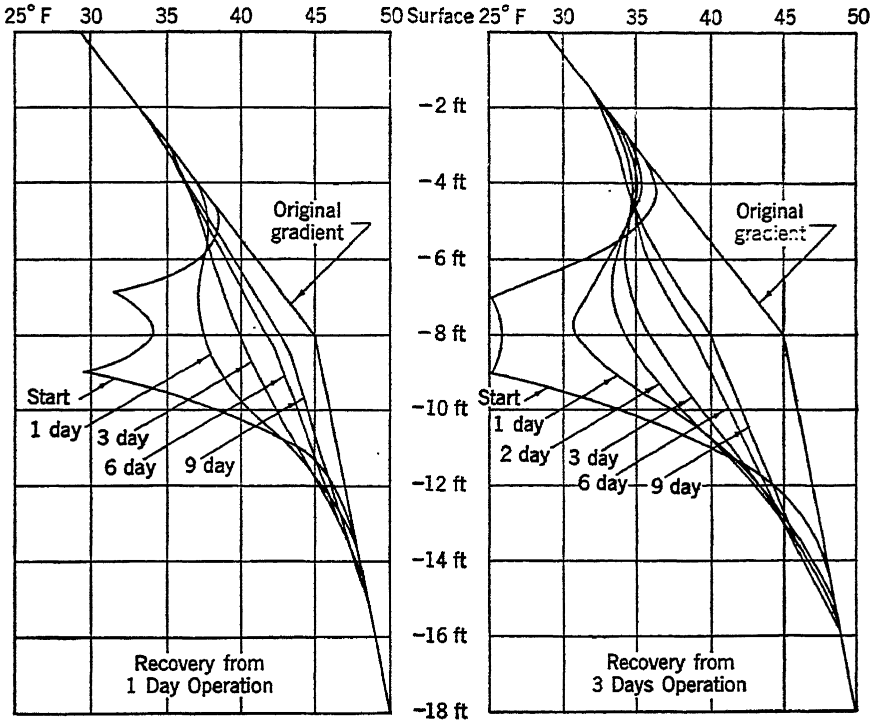

FIG. 7-4. GRADUAL RECOVERY OF GROUND TEMPERATURES.

easily have been found for any other time intervals and for any other total time. Three sample temperature fields are shown in Fig. 7-3. To find the twelve sets of temperature gradients, eight man-hours were required for one inexperienced in using a standard calculating machine. To find the corresponding temperature fields, for the simplified ideal case, required one hundred man-hours when Carslaw's analytical solution was used.

It was also desired to find the changes that took place as a function of time in the established temperature fields when the heat sinks were eliminated and *the earth was allowed to recover toward its initial temperature distribution.* This problem has not been attacked analytically anywhere in the literature. The method of numerical solutions gave the answer to this problem very readily. The results are shown in Fig. 7-4.

CHAPTER 8

SUCCESSIVE CORRECTIONS — A PATTERN OF THOUGHT

Frank Baron *

Synopsis. Similarities that occur in various numerical procedures of analysis are inspected and discussed. These include similarities in physical conditions and in the fundamental groups of requirements which must be fulfilled in several fields of mechanics. Analogies which result from similar requirements to those of statics and geometry in structural mechanics are considered. Various numerical procedures are classified according to the order observed in meeting controlling requirements. Such procedures also have been classified as to the sequence followed in making computations. They have been analyzed and compared in the fields of structures, vibrations, and elasticity. Additional examples are given in other fields of interest. A discussion is included of preferences that may exist in selections of initial estimates or approximations and of sequences of computations. The relation of structural analysis to structural behavior is indicated.

Correspondence of Criteria in Several Fields

Much similarity in thought is evident in various numerical procedures of analysis. Included are similarities in physical conditions or in fundamental groups of statements that must be satisfied for the different fields of mechanics. Analyses of physical phenomena often consist of fulfilling requirements of two independent groups of statements henceforth referred to as (*a*) and (*b*). These requirements are usually interrelated through another set of conditions or physical constants (*c*). Such requirements and interrelations occur frequently in Newtonian or causative mechanics.

Requirements to be Met in Structural Mechanics. These groups of statements are well illustrated in the field of structural mechanics. The same requirements occur in the theories of elasticity, plasticity,

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and buckling. In general, structural analysis deals with the following sets of conditions: (a) *Statics.* For any structure or part of a structure, forces and moments must be in equilibrium. (b) *Geometry.* Geometrical fitness of elemental parts requires no unwarranted linear or angular discontinuities at any station or about any closed circuit. This may be stated in symbolic or mathematical form, and the resulting expressions are commonly called equations of compatibility. (c) *Statics and geometry are interrelated through the known or assumed properties of materials.* Such relations may be linear or nonlinear. The above conditions may be modified to include Newton's statement relating force with the geometry of motion and of the corresponding properties of materials for dynamic conditions of loading.

Such groups of statements serve as bases for devising procedures of analysis. Generalized statements or conditions, however, may have to be restated more specifically for purposes of organizing the sequence of computations. Those who have had an interest in convergence procedures recognize that such examinations and restatements are at times difficult matters. Examples will be given where the fundamental requirements have been restated for particular classes of problems.

Steady Flow in Conductors.[1] The following sets of conditions control studies of the steady state of flow of fluids or of electric current in networks of conduits or conductors: (a) *Continuity of flow.* The total inflow at a junction must equal the total outflow. (b) *Continuity of potential.* The total change in potential about any closed path is zero. (c) *In addition, loss of head and quantity of flow are related through known or defined physical constants.* These include the properties of conductors and of the fluid. Similar statements apply to studies of the steady state of heat flow in thermal conductors. The concepts (a) of continuity of heat flow and (b) of the total change in temperature about any closed path are unchanged. Other constants (c), however, are employed to relate flow and drop in temperature.

Two-dimensional Flow. The requirements for continuous or two-dimensional conductors and idealized fluids may be stated symbolically. They result in Laplace's partial differential equation, namely,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (8\text{–}1)$$

Additional examples may be given of problems in two-dimensional space governed by Laplace's equation. These include studies of elec-

[1] See page 137 for all references to Chapter 8.

trostatic and magnetic fields, the steady state of direct current in flat plates, and seepage through earth masses.

Torsion of Multiconnected Cylinders.[2] The following requirements and interrelations are observed in problems of the torsion of multiconnected thin-walled cylinders: these cylinders may be square, triangular, hexagonal, or of any cross section that will "nest" together. (*a*) *Statics.* The quantity of force f "flowing" into a junction must equal the quantity of force f "flowing" away from a junction. (*b*) *Geometry.* Continuity of deformations along a closed path must be observed. This requirement plus the usual assumptions lead to an expression commonly called Bredt's formula. (*c*) *The relation between stress and strain* is already included in Bredt's formula given below.

$$\oint \frac{f}{t}\, dL = 2G\alpha A_c \tag{8-2}$$

that is, the product of the constant $2G\alpha$ and the area A_c within a closed circuit must equal the algebraic sum of the products of the unit shears f/t and the increments of length dL along the closed circuit. The constants G and α are the modulus of elasticity in shear and the angle of twist per unit of length, respectively.

The above statements may be modified to include examples of similarly shaped cylinders subject to flexure only. In this case, an excess or deficiency of shear force at a junction may be considered, and also a zero value for the angle of twist per unit of length.

The example of multiconnected thin-walled cylinders subject to torsion may be studied by use of a stress function, a soap film, or a membrane. Rigid plates whose surface areas are identical with the cross-sectional areas of the cylinders are imagined to be suspended over the hollow areas of the cylinders. In addition, taut membranes whose horizontal dimensions are defined by the wall-thicknesses of the cylinders are considered as being attached to the plates. An inflation pressure proportional to the angle of twist of the cylinder is considered to be applied. Similar groups of restrictions apply here as elsewhere: (*a*) *Statics.* The total force on each plate due to the inflation pressure must be balanced by the tensions of the membranes. (*b*) *Geometry.* The transverse slopes of the membranes must be in agreement with the respective elevations of the plates. This requires for the case of equal wall thicknesses, a "balance" of the transverse slopes at junctions of the polygonal cylinders. (*c*) *Conditions of statics and geometry are interrelated by the accepted properties of the membranes and plates.*

It is to be noted that the preceding requirements of (*a*) statics and of (*b*) geometry have been interchanged here with those of (*b*) geometry and of (*a*) statics, respectively. As stated elsewhere, members with a solid cross section may be studied by dividing the section into a network or grid.

Example of a Riveted Joint. An example of a joint subject to a longitudinal tension and consisting of several plates connected by a longitudinal line of rivets is considered. An "elastic theory" is assumed, *for discussion only,* to describe adequately the behavior of such joints. The structural action of such joints, however, may not conform with the conditions of an "elastic theory." But based upon elasticity, the following requirements may be stated: (*a*) *Statics.* Forces must balance for any isolated part of the joint. For example, the isolated parts may consist of a series of plates bounded by parallel sections transverse to the longitudinal axis of the joint or of a junction of a plate with a rivet. (*b*) *Geometry.* Geometrical conditions require that the total change in longitudinal deformations about a closed path be zero. A convenient path to consider consists of adjacent plates bounded by adjacent rivets. The contributions to geometrical movements consist of changes in length of plates and shear deformation of rivets, since in an "elastic theory" the slip of plates relative to each other is usually neglected. (*c*) By this theory the change in length of a plate between adjacent rivets is considered to be proportional to the force in the plate, the distance between adjacent rivets, the cross-sectional area of the plate, and the elastic properties of the material. In addition, the shear deformation of a rivet between adjacent plates is considered to be inversely proportional to the resistance of the rivet to such deformation.

Interchange of Relations between Fields

Analogies. Membrane analogies and stress functions are often employed as aids in obtaining solutions to problems in the theories of elasticity and plasticity. Analogies also have been devised in the fields of electricity and magnetism, heat flow, and fluid flow. In general, analogies occur among different fields as a result of the same requirements for similar groups of controlling conditions (*a*) and (*b*) and identical forms of relations (*c*) between these groups. Consequently, interchanges of measurements or computations between related fields of mechanics are possible. In the same way, similarities in the requirements of groups (*a*) and (*b*) of a given field of mechanics make sub-

stitutions of related computations possible within that field itself. Differences may then occur in terminology, physical nature of the problem, and quantities involved, but not in concept.

Electrical Analogy for Structural Vibrations. An example may be given of the analogy between simple mechanical structures and electrical circuits. In this analogy, mass and inductance, spring stiffness and capacitance, force and voltage, take identical roles.[3] In the same way damping may be related to resistance, displacement to condenser charge, and velocity to current. The dual analogy, with an interchange of mass and spring stiffness, force and displacement, is also used. The latter interchange is useful in studies conducted solely of mechanical systems. It is reported elsewhere to be useful in substitutions of measurements in the related fields of mechanics.

Interchange of Statics and Structural Geometry. Similar relations and possible interchanges may be illustrated in the field of structural mechanics.* Computations of shears and moments are defined in the same way as those of slopes and displacements in problems of geometry dealing with minute movements. The contribution of a small angle change (curvature or minute rotation) to a displacement is computed as the product of the angle change and the related distance. By definition of such computations, (*a*) an angular displacement (rotation) about an axis may be interchanged with a force in the direction of the axis of rotation, and (*b*) a linear displacement may be interchanged with a moment about the axis of displacement.

Statics requires a balance of forces and of moments. Conditions of geometry about a closed path require a balance of angular and linear displacements. Thus, computations of statics and computations of geometry are interchangeable. (This general concept was developed by Professor Hardy Cross. It may be of interest to compare these statements with those of a vector algebra.) Such relations have resulted in the conjugate beam method of computing movements, the conjugate arch, a hydrostatic analogy, a column analogy, and the more informal statement of the pressure-line concept.

Generalization Beyond Structures. In general, the above relations may be extended to include the study of curves or surfaces where the units of the ordinates differ from those of the abscissas. The scales are then independent of each other and the approximations that apply with small angle changes are acceptable in computations of geometrical

* Discussion of the various theorems of reciprocal relations commonly credited to Maxwell, Muller-Breslau, Betti, and Rayleigh is omitted.

relations. Many such curves and surfaces occur in the different fields of mechanics. Several examples may be cited: shear and moment diagrams, displacement-time curves, velocity-time curves, acceleration-time curves, stress-strain curves, and force-displacement curves. *Geometrical relations for these and for many other curves may be computed as corresponding statical relations of imagined beams.* The curvatures, slopes, and displacements are treated as imagined loads, shears, and moments, respectively. These relationships may be formalized in the same way as other procedures of analysis. Such formalization includes an automatic system of signs and an orderly procedure for recording computations. An example of such interchange is indicated in Table 1, in which a displacement-time curve is considered.

TABLE 8–1. BEAM ANALOGIES

Displacement-time curves	Analogous beams
t = time	
$t_f - t_i$ = total change in time	L = span length
dt = increment of time	ds = increment of length
y = displacement	m = moment
dy/dt = slope = velocity	dm/ds = v = shear
d^2y/dt^2 = curvature = acceleration	d^2m/ds^2 = dv/ds = w = load per unit of length
terminal displacements	end moments
terminal velocities	end shears

Geometry of motion. The accelerations during a total change in time must be "balanced" by the terminal velocities and displacements.
Geometry of small angle changes. The angle changes for any closed path must balance.
Statics. Forces must balance.

Statements similar to those at the bottom of Table 8–1 apply to many other relationships in mechanics. The writer has found such interchange of criteria and of computations useful at times in studies of other curves and relationships. Certain total and partial differential equations,* — several problems in structural vibrations and in the kindred subjects of kinetics and kinematics have been inspected in this way. In addition, convergence procedures devised for problems of structures have been extended to studies in these related fields.

* An angle change is related to the operators d^2/df^2, $\partial^2/\partial f^2$, $\partial^2/\partial f_1\partial f_2$. Total differential equations and partial differential equations involving two independent variables may be interpreted as defining the geometrical requirements of curves and deflected surfaces, respectively.

It may be noted that the geometrical qualities of certain stress-functions have been related previously to corresponding problems of statics.[4]

ORDER OBSERVED IN SATISFYING REQUIREMENTS

As stated previously, solutions to problems of structural mechanics must satisfy conditions of statics and geometry which are in agreement with the properties of materials. Hardy Cross has stated that engineers have tried continuously to combine successfully these conditions with existing or conceivable methods of analysis. The various "methods of analysis" are identical in the above respect, and differ only in the sequence and arrangement of computations, and in the language employed. The language may be (*a*) pictorial or graphic, (*b*) symbolic or algebraic, (*c*) arithmetical, or (*d*) oral. A pattern of thought may be observed in the numerous solutions that have been obtained.

Subdivision of Methods. Formalized procedures of analysis may be classified for convenience into one of the following groups as suggested by H. M. Westergaard:

(*A*) A statically possible solution is first considered and is then checked for errors in geometry. The method selected for checking errors does not make various procedures of analysis essentially different from each other. Independent patterns of statically possible corrections are chosen and combined with the initial estimate of structural action. The combination must meet prescribed requirements of geometry irrespective of the procedure used in effecting the combination. If an algebraic approach is used, the resulting equations are those of geometry, the unknowns having been previously interrelated through statics and the properties of materials.

(*B*) As another approach, an initial solution meeting the prescribed requirements of geometry is selected and checked for possible errors in statics. Corrections meeting the requirements of geometry are chosen and combined with the initial solution. In this case, algebraic procedures result in equations of statics, the unknowns having been previously interrelated by geometry and the properties of materials.

Energy Relationships. (*A′*) Procedures based on obtaining a minimum of energy by variation of stress may be inspected in the same way. Statics and geometry are interrelated by the definition of strain energy. A statically possible distribution of forces is observed con-

tinuously as the strain energy is varied. The requirement of a minimum of strain energy is a requirement of geometrical fitness.

(B') Energy procedures that consist of obtaining a minimum energy by variation of shape have similar characteristics. Statics and geometry are interrelated by the definition of a change in total potential energy, energy due to position of load, and strain energy. In this case, requirements of geometry are continuously considered as shape is varied. Conditions of equilibrium require a minimum of the total potential energy of the structure and its loads.

Sketching Based Upon Judgment. Other sequences of computations may be used in fulfilling requirements of statics and geometry. These include procedures consisting of *judicious guesses* and the necessary checks. Procedures also exist in which errors in fundamental groups of conditions are successively adjusted by developed techniques of sketching. Such techniques are powerful tools of analysis to those who possess the art of sketching. This art has been used successfully in studies of flow nets in different fields of mechanics.[5] It is particularly useful in preliminary analyses for design and in the interpretation of structural behavior.

Examples of such procedures include sketches of deflected structures[6] and the concept of the pressure line.[7] In the former procedure, sketches of deflected structures and of corresponding moments are successively adjusted. An acceptable solution is obtained when the distribution of moments is in approximate agreement with the angle changes per unit of length and with the properties of the material. In the latter procedure, statically possible solutions are successively revised; requirements of geometrical fitness are checked after each revision.

Numerical Convergence. Many procedures of numerical convergence can be classified into the same categories as those listed above for formalized analyses. In general, convergence procedures consist of a series of approximations, checks of possible errors, and adjustments of errors. Differences, however, occur in the order observed in meeting the fundamental groups of requirements. In structural analysis, solutions have been devised continually meeting the requirements of geometry and successively adjusting the errors in statics. *Moment distribution described in Chapter 1 is one example.*[8] Other procedures have reversed this order, for example, *the distribution of angle changes at the joints.*[9] Procedures have been devised, also, in which the initial estimate satisfies requirements (a) or (b) but with

errors in the other. These errors (say in (*a*)) are corrected, disturbing, however, the requirements of (*b*). Succeeding computations alternately correct the errors introduced to one group by the corrections to the other group.

Geometrical Requirements Preserved in Numerical Analysis. In moment distribution the initial estimate of moments is made by considering a solution consistent with the requirements of geometrical continuity. In the case of no translations of joints, all members of a frame are considered to be initially without rotations at their ends. The errors in statics occurring at each joint are successively corrected until an acceptable balance of moments is obtained at each joint. The patterns of corrections, however, preserve the requirements of geometry throughout the entire procedure of analysis.

A seemingly endless number of extensions of this procedure have been devised permitting any combination of joint rotation with joint translation. These extensions, however, have the common characteristic that at each stage of the procedure the selected patterns satisfy requirements of continuity. They may differ in the choice of initial patterns, in the sequence chosen for successively correcting the errors in statics, and in the elemental parts selected for checking such errors. For example, the selected elements may consist of joints of planar or nonplanar structures, of closed circuits or panels, or of structural parts lying between parallel sections.

Adjustments to Meet Statics. Errors in statics may occur either in the balance of forces, the balance of moments, or of both. Unbalance in moments may occur at joints of structural frameworks, or along edges or supports of continuous slabs.[10] Unbalanced forces may be illustrated by unbalance of cable tensions at the tops of suspension bridge towers, of horizontal thrusts and moments at the ends of arches on elastic piers, by deficiencies of shear in panels of open-web girders, or of horizontal shears between floors of building frames. In addition, errors may occur in the balance of forces at joints of multiple intersection trusses or at intersections of imagined lattices, grids, or networks approximating continuous media. Several procedures devised for solving problems of plane stress belong in the latter grouping.

Differences between numerical methods may be observed in the sequences that have been suggested for successively correcting errors in statics. Direct and indirect methods of analysis have been devised. In addition, different patterns and sequences are possible for cases where forces only, moments only, or both are successively balanced.

In each of these cases, displacement patterns are considered in which requirements of geometry are always satisfied.

Requirements of Statics Preserved in Numerical Analysis. Many related procedures have been presented wherein the initial guesses and successive corrections preserve the requirements of statics. Corresponding conflicts of geometry are inspected and successively corrected until an acceptable degree of geometrical fitness is obtained. As stated previously, geometrical fitness requires *no unwarranted linear or angular discontinuities at any station or about any closed circuit.* Differences between numerical methods may be observed again in the choice of initial guesses, patterns of successive corrections, and in the circuits or elemental units selected for checking errors in geometry. Illustrations may be given of analyses where successive adjustments have been made of angular discontinuities over the supports of continuous girders, of linear discontinuities in members of multiple intersection trusses, and of linear and angular discontinuities in adjacent closed circuits of structural frames.[11] In addition, linear discontinuities of imagined lattices or networks approximating plates or other continuous media have been successively adjusted.

Examples in Other Fields. Membranes and certain stress functions in the theory of elasticity have been studied as problems related to the geometry of deflected surfaces. Partial differential equations involving two independent space variables may also be interpreted as defining the geometrical requirements of deflected surfaces. Examples are Laplace's and Poisson's equations in two dimensions, Prandtl's stress function, and the equations of membrane analogies for problems of torsion and for fluid flow. Additional examples include Airy's stress function, Nadai's stress function for slabs, and the related stress function of Carothers for slices. The deflected surface must meet the requirements of a governing equation and the boundary conditions at one or more curves limiting the extent of the region. *Desired variables are usually related to the displacements, slopes, or curvatures of the membranes or of the stress functions.*

Lattices. Several numerical procedures have used imagined lattices, networks, rosettes, and taut strings *to approximate the desired deflected surface.*[12] Such procedures are often related to problems of structural analysis. A close inspection is usually needed of the geometrical requirements at the boundaries or over prescribed regions. In addition, the required loading at intersections of lattices or over elemental areas must be examined. Solutions are then obtained for the deflected

lattice work which satisfy conditions of statics and of geometry. *Forces for any isolated part of the imagined lattice must be in equilibrium, and geometrical fitness at any station or about any closed circuit must be obtained.* In one such type of analysis the selected displacement patterns continually satisfy conditions of geometry, and the unbalances in forces are successively adjusted. Procedures also exist in which statics is continually satisfied and the geometry of displacement is successively corrected.

Problems of closed traverses involving requirements of geometry alone have been studied in the same way. Conditions of "balance" for linear and angular displacements have been considered as independent requirements interrelated through the definition of an angle.

Sequences of Computations Used in Analyses

Some distinctions exist in analyses that may be classified for convenience as procedures of successive approximations, corrections, judicious guesses, iteration or step-by-step procedures, and total corrections made in a single step. These distinctions depend on the sequence of computations used in obtaining solutions.

Successive Approximations. Solutions may be obtained by adding a correction to an approximation and treating the revised answer as a new estimate of physical behavior. The procedure has the advantage that arithmetical errors are not cumulative and, if made, are observable and ultimately eliminated. In addition, if the convergence is unsatisfactory it may be improved with a little judicious guessing. Immediate scale is often obtained of quantities involved and the final answers sensed before the computations are concluded. *The procedure lends itself to studies of nonlinear relationships.*

As an example consider the buckling of structural members and frameworks. A deflected structure is assumed and a revised estimate of the deflected shape is obtained from computations making use of moments consistent with the initial shape. This is possible when the relation between moment and angle-change per unit of length is known for various degrees of axial load. *This sequence of computations is repeated until statics and geometry are in agreement or until total yielding is reached.* A similar procedure has been used in studies of the natural modes of vibrations of mechanical and structural systems and is usually credited to Stodola.[13]

Several differential equations have been studied by the writer and a numerical procedure devised in which a curve is first approximated

s a solution. The approximation is investigated by use of the governg equation and a solution is obtained through a sequence of successive pproximations. This procedure may be studied mathematically, and leads to solutions expressed in series form.

Successive Corrections. Instead of computing revised totals after ach pattern of corrections, it may be expedient to correct only the rrors introduced by the preceding set of corrections. A sequence ay be chosen wherein each initial error is corrected by an amount qual to the error. The individual pattern of corrections, however, eeds to be inspected as these revisions may introduce a new set of rrors elsewhere. Each set of errors is successively adjusted and disibuted as previously. Convergence may be helped at times by a idicious choice of over-corrections.

These procedures may have the advantages of dealing with smaller numers and of not repeating certain computations. Orderliness in the elected sequence of computations and in recording them is usually nportant. In addition, arithmetical errors should be readily observble and easily eliminated. The procedure may be made one of sucessive approximations by stopping the distribution at desired stages nd obtaining totals.

Many examples of distribution procedures could be classified here, ncluding those of successive relaxation of constraints. Successive reaxation of constraints may be interpreted at times as successive corections of errors in statics.[12] The procedure has been used also in tudies in the theory of elasticity. Solutions to some problems have een devised by additions of stress-functions which successively corect certain errors introduced by the preceding function. In addition, here are methods for obtaining solutions to linear simultaneous quations [14] and to certain total differential equations that are of this haracter.

Successive Guesses Based upon Judgment. An important proedure of analysis consists essentially of a series of guesses based upon udgment or experience. Judgment is checked against the required onditions of the physical problem. If the requirements are met, then bviously the guess is correct. If the initial estimate is inaccurate, nother guess may be made and checked. Each guess is tempered by he judgment of the computer, the preceding set of computations, and he need for making such computations. The procedure may be rude, informal, and may not lend itself to standardization. These haracteristics do not necessarily detract from the value or the power-

fulness of the procedure. It may be noted that the procedure permits the full exercise of judgment and a continuous close inspection of the physical nature of a problem. It also lends itself to the further development of judgment. It is an important tool of analysis in the hands of a designer. An approximate answer is often all that is needed in checking designs or comparing structural types.

Examples of the Use of Judicious Guesses. This procedure has been used in computations of natural frequencies of undamped vibrations, and also of displacements of forced vibrations. It is then usually credited to Holzer.[15] The same sequence is used in the concept of the pressure line. An additional example may be mentioned of a graphical procedure for analyzing continuous girders. The procedure has been presented in textbooks of graphic statics. It is suggested that a guess first be made of a solution satisfying statics. Then a deflected structure should be drawn formally which is in agreement with the resulting bending moments and the properties of the material. It is further suggested that successive guesses be made until the geometrical picture is considered acceptable.

Iteration or Step by Step. These procedures are similar to numerical methods of integrating a differential equation. In numerical methods of integration a curve is traced. The value at each successive station is defined by its relation to the preceding step. Many problems in mechanics may be studied in the same way. It is preferable in these cases to consider the physical nature of the problem. In addition, it is important to define the interrelation of the variables.

The procedure of *iteration* is illustrated by an example in the study of vibrations; namely, a mechanical system with a single degree of freedom and with known conditions at a particular time. The changes in geometry with respect to time may be obtained through Newton's statement relating force and acceleration if the exciting force, damping characteristics, and the accepted properties of materials for static and dynamic conditions are known. By considering small increments of time, each successive force, deformation, displacement, velocity, and acceleration may be computed.

The procedure of *step-by-step* analysis may be illustrated with the usual graphical procedure for tracing a deflection curve of a loaded beam on simple supports. Values of the angle changes along the beam are considered to be known. *An initial estimate is made of the rotation at a support and then, step by step, the curve is drawn between supports.* If needed, the initial estimate of the end rotation is corrected. A

numerical procedure, however, may be substituted for these graphical computations.

Corrections Made in a Single Step. Certain problems in structural analysis lend themselves to a procedure consisting of an estimate and a single set of corrections to the initial estimate. An obvious example is that of the analysis of a structure with but one degree of statical indeterminateness. Other examples include numerical procedures consisting of established or tabulated computations for a single installment of corrections. These procedures are illustrated by the column analogy, the hydrostatic analogy, and the neutral point method.

The concept of adding a single correction to a first approximation has had other uses, however. It lends itself to qualitative and quantitative studies of certain forms of questions raised by engineers. It is used often by designers of structures in their preliminary studies of structural behavior. For example, computations may be made at times by some convenient theory which includes the effect of important variables. In addition, the simple theory may yield an immediate scale of quantities involved. If desired, a correction may be added to include the effect of neglected factors or to extend the results into those of a more "exact" theory.

Single Corrections to an Approximate Theory. The procedure of computing results through a simple theory of mechanics and adding a correction to extend the results into those of a theory of elasticity or of a theory of plasticity is frequently useful. The writer has used this concept in correlating the effects of plasticity with an elementary theory of mechanics.[2] These effects are computed first through the ordinary theory of mechanics. Then a single pattern of corrections is added to extend the results into those of a theory of plasticity. For example, the stress s in a fiber of a cross section subject to an axial load and a moment is computed as

$$s = s_o + s_c \tag{8-3}$$

where s_o is the stress defined by the ordinary theory of mechanics,

$$s_o = \frac{P}{A} \pm \frac{Mc}{I} \tag{8-4}$$

and s_c is a correction stress to account for the nonlinear stress-strain relationship of the material. Several problems in the theory of elasticity have been studied in the same way. Differences in results of the theories then appear as corrections to the simple theory of mechanics.

Other Influences upon Choice of Method. A discussion of advantages among numerical procedures has been given. Objections may also be cited, as some procedures do not lend themselves to studies of certain forms of questions. In addition, certain convergence procedures may not be practical unless required constants have been standardized or are easily available. The following factors also influence choice:

Order. Selection may depend at times on the order preferred in meeting some requirements and adjusting the errors in others. In structural analysis, if relations in statics such as moments, forces, or stresses are of first interest, procedures continually satisfying geometry are usually preferable. It may be desirable to use procedures continually satisfying statics if displacements are of first interest.

Sequence. The selection of a sequence of computations may depend on speed of convergence, ease in recording and remembering preceding computations, and elimination of repetition. The choice may depend at times on the possibilities of revising the sequence itself, of introducing variations in dimensions or in properties of materials, or in a simple physical interpretation of numerical terms.

Interpretation of Structural Action. The nature of analytical studies or numerical procedures may depend on the interpretation of structural action, as well as the initial estimate of dominating characteristics of structural behavior. This may be illustrated by comparing the behavior of the following structural types: (*a*) A rectangular frame with large values of depth-to-length ratios of all members, as contrasted to a frame with small values. The effects of continuity may be a small factor in the former case. (*b*) Tall building frames with large values of column-to-girder ratios of stiffness and those with small values. Comparisons of structural action for extreme cases may well be of particular interest in studies of wind stresses. For example, it may be desirable to study for each extreme case the effects of slight changes in the ratios of relative stiffnesses on the values of the lateral movements. (*c*) Vierendeel girders with different values of depth of top chord to depth of bottom chord. In addition, it may be desirable to compare the behaviors of such types when the columns are either extremely stiff or flexible relative to the chords.

Primary Stresses and Deformation Strains. The nature of analytical studies also depends on whether an effect is due to a source of stress or a source of strain. A natural distinction exists between these sources and their effects. Primary stresses in a structural member are a result

of load. Deformation strains are a consequence of internal or external movements of the structure. Examples of sources of strain are abutment movements, differential settlements, shrinkage, temperature changes, and residuals. Secondaries or participation strains resulting from movements caused by primary stresses or by interaction of members are also included. Thus, load dominates as a source of stress, and the geometry of movement as a source of strain. Methods of analysis may be chosen dealing directly in terms of statical relations, or in terms of geometrical relations to fit the source of structural action.

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CHAPTER 9

NUMERICAL METHODS OF ANALYSIS OF BARS, PLATES, AND ELASTIC BODIES

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Scope of Text. This chapter is a survey of the approaches to numerical or arithmetical methods of stress analysis as distinct from the more formal analytical methods. The general philosophy and basic techniques of iterative procedures, step-by-step methods, relaxation, and continuity restoration are considered in relation to a variety of problems, and attention is given to means for obtaining increased accuracy with numerical methods. Some of the methods described have been developed by the writer and his associates in the Engineering Experiment Station of the University of Illinois as part of the work on cooperative research projects sponsored by the Office of Naval Research, the David Taylor Model Basin, the Public Roads Administration, and other agencies.

Approaches to Numerical Methods of Analysis

Distinctive Features of Numerical Methods. *"Numerical" methods, as distinct from "analytical" methods, deal with the numerical values of the significant variables at all stages of the computation.* Formulas expressing the dependence of one variable on others are consequently not obtainable generally by these methods. On the other hand, variations in material properties and boundary conditions that are often impossible to consider by formal methods may usually be considered quite readily by the cruder but more adaptable and flexible numerical methods.

In many respects numerical procedures are like experiments: to find a functional relationship, for example, one may have to solve a series of problems and examine or plot the results. The dangers involved in such a process are obviously great, and the required amount of calculation may be enormous. However the new computing devices now

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being developed make it possible to carry out calculations that heretofore have offered almost impossible difficulty. Methods that in the past might have been too lengthy for consideration may yet prove to be the most satisfactory for high-speed digital computers.

Iteration and Step-by-Step Calculation. Numerical methods may in general be divided into several types, and obvious distinction can be made between those based on mathematical concepts and those based on a physical action or analogy. However, these groups overlap. From the mathematical point of view the following are fairly clearly separable:

(*a*) *Iterative Methods* — in which an assumption is made of initial values, reasonable or otherwise, of the significant variables, and these values are then modified systematically until a stable set of values is obtained. These methods have the advantage that in many problems a reasonable starting point is available, and the corrections are relatively small and quickly carried out.

(*b*) *Step-by-Step Methods* — in which a start is made at some known point and values are computed successively in steps from this point.

In general these methods are similar to procedures for the solution of sets of simultaneous equations. However, the equations need not actually be written down in order that the methods be applicable.

Physical Methods and Analogies. From the physical point of view a classification can be set up in terms of the methods and concepts of stress analysis of a structure:

(*a*) *Relaxation* — in which the structure is at all times maintained in a continuous state, but statical equilibrium is gradually achieved by "liquidating" undesired "residual" forces by patterns of joint displacements.

(*b*) *Continuity Restoration* — in which the structure is at all times maintained in a state of equilibrium, but certain internal (or external) discontinuities are permitted. These are gradually liquidated by suitable statically self-balanced systems of forces without affecting the equilibrium.

(*c*) *Trial Loads* — This procedure which has primarily been used for arch dam analysis is a variant of the continuity restoration scheme in which the discontinuities are eliminated as much as possible by selecting component loads (for arches and cantilevers) from a previous study of such loadings.

These methods may be used in other problems than the analysis of a structure, in which case the various aspects of the methods may be

considered as analogies. These procedures generally fall into the category of iterative methods, but they may equally well be used in step-by-step analyses. For example, the well-known method of moment distribution, due to Hardy Cross, is a special type of relaxation procedure, and the ordinary use of the method involves the iterative principle. However, the so-called "direct distribution" methods are essentially step-by-step procedures.

Variational Procedures. Finally, a distinction must be made between those methods which *involve operations on elements, and those which involve operations on the whole.* The methods described in the previous paragraphs are all of the former type. The latter type include methods such as energy methods and procedures involving the minimizing of certain functions of the errors in a set of equations. Important examples of such "variational" methods, in addition to the methods of minimum potential energy, and the method of minimum complementary energy,[1] are:

(*a*) *Collocation* — where the errors are made zero at a number of arbitrarily selected points only.

(*b*) *Least squares* — where the integral of the square of the error is made a minimum, and

(*c*) *Other methods,* including weighted least squares, Galerkin's method, etc.

These methods do not necessarily involve numerical procedures, but they can be used numerically and they can often be related to the more direct numerical processes. Also, the technique of relaxation can generally be used with these methods.

Types of Problems

"Discrete Joint" Problems. In problems having a finite number of degrees of freedom, where values of a significant variable are to be determined only at discrete points or "joints," the solution has a direct mathematical counterpart in terms of a set of simultaneous equations which may or may not be linear. So-called "exact" solutions may be obtained for these problems if the equations are not too complicated. The form of the equations determines the type of numerical procedure best adapted to the problem. The only approximation involved in the solution is that inherent in the numerical process used; in general, convergent processes may be carried to as great a degree of accuracy

[1] See pages 167–168 for all references to Chapter 9.

as desired. The analyses of frameworks or assemblages of bars with or without flexural resistance fall into this classification. For convenience, such problems are described here as "discrete joint" problems.

In general, the ordinary numerical methods are adaptable only to discrete joint problems (including analogies), and the variational methods to problems involving continuous values of a variable, or "continuum" problems. However, numerical methods may often be used in variational procedures.

"Continuum" Problems. Problems in which a variable is to be determined over a whole region generally involve differential or integral equations which may or may not be linear. These "continuum" problems in certain cases may be solved "exactly" by means of the variational procedures, but the strictly numerical methods cannot be applied directly. Two principal techniques are available for the application of the ordinary numerical methods to continuum problems:

(*a*) *Mathematically,* by the substitution of finite differences for derivatives, or in general, by the approximation of the continuum problem with a discrete joint or nodal system.

(*b*) *Physically,* by a lattice, framework, or other structural or mechanical (or electrical, etc.) model. This physical analogy to the actual problem is not an exact one, but it may be made as accurate as desired except where discontinuities or singularities are involved.

The two techniques are related; it is often possible to develop a mechanical model for which the exact solution leads to the same equations that are obtained by means of the finite difference procedure applied to the continuum.

Articulated Models of Beam. Consider, for example, the determination of the deflection of a beam subjected to a given loading. The ordinary first-order finite difference substitution for the derivatives leads to a difference equation analog for the differential equation of the elastic curve of the beam. However, precisely the same difference equations are obtained for the rigorous analysis of a structural model that resembles the given beam. The model is made up of perfectly rigid links connected by deformable hinges. The amount of rotation at a hinge is proportional to the moment at the joint and inversely proportional to the values of modulus of elasticity and moment of inertia of the beam at the point. Loads are considered to be applied only at the joints.

Articulated Model of Plate. An extension of this model leads to a representation of a plate. The physical model for the plate is shown

in Fig. 9–1. In order to arrive at the same equations which are yielded by the difference equations approach, coil springs are introduced in the model to carry the twisting moments. The upper part of Fig. 9–1 shows the make-up of the model at a free edge of the plate. The rigid links are only half as wide as elsewhere, and the coil springs on the

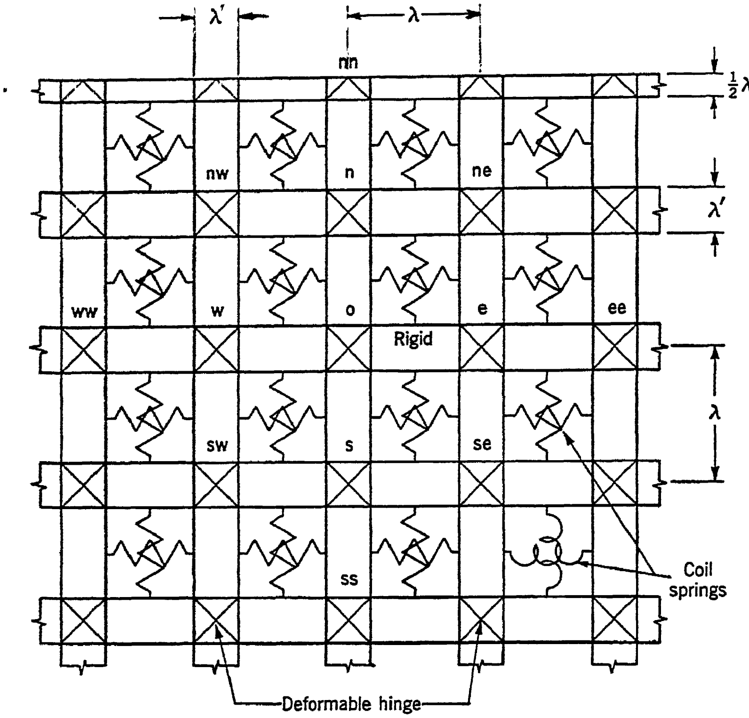


FIG. 9–1. PHYSICAL MODEL LEADING TO DIFFERENCE EQUATION ANALOG FOR A PLATE.

outer surface are omitted. This model leads to consistent relations at all joints, including those at the edge of the plate, and its analysis gives precisely the same equations as those derived from finite differences.[2] The use of the model offers certain advantages: there is no ambiguity concerning the conditions at a boundary; *statical checks on the results have a physical meaning and can be made more accurately;* variations in dimensions and physical properties can be more easily treated. However, it is not convenient to develop a model which represents the same conditions as a higher order difference equation analog.

Similar models can readily be derived for other problems. A network of strings, for example, leads to the difference equation analog for a membrane. And a model similar to that shown in Fig. 9–1 with

the coil springs replaced by a linkage to resist shearing stresses, leads to the difference equation analog for Airy's stress function for plane stress or strain.

Probable Errors. It must be emphasized again that the numerical analysis can be made as accurate as desired for the analogous problem, but some error is always involved in representing the continuum by a discrete set of points. This error can be minimized in various ways, some of which are considered in detail later: (*a*) by taking more elements or joints; (*b*) by extrapolation from several solutions for the same problem involving different spacing of joints; (*c*) by modifications of the original data, or by modifications of the computed results to correspond more closely with relations applying to the continuum. *However, probably the best approach when it can be followed is to use the numerical procedure only for the purpose of computing a correction to or a variation in a standard solution.*

Independent Variables. The difficulty of solution of a problem increases materially with the increase in number of independent variables. For example, the stress analysis of a bar, rod, or shaft, even for instability problems or for steady state vibrations, is a fairly simple matter. Any of the procedures described may be used for such problems with different degrees of accuracy and speed, but in general they all lead to solutions in a reasonable length of time, and the solutions can be made fairly accurate.

However, problems involving two variables, such as plates or plane stress and strain problems, or problems involving the transient response of a string, bar, or shaft to an impulsive loading, in which time is the second variable, are much more difficult. In these problems, the relative usefulness of the different methods shows a greater spread: in some of the cases certain methods do not converge to a solution at all. However, such problems have been successfully solved by both numerical and variational procedures.

Three-Dimensional Problems. The next stage in difficulty are problems involving the time response of a two-dimensional system or the static behavior of a three-dimensional body. These problems are extremely difficult and *few have been successfully attempted by numerical means.* Certain special problems which involve symmetry with respect to one or more variables can be reduced to a problem involving only two independent variables and have been successfully attacked.

Techniques for Reduction of Number of Variables. Under certain conditions the number of variables which are required to describe the

behavior of a body may be reduced, permitting a simpler treatment of the problem. In addition to those problems in which considerations of symmetry permit a reduction in the number of variables, there are others which are of the type where a particular form can be assumed for the solution. In this class of problems one may also include those in which the *variables are separable.* This is the case in general for steady state harmonic vibrations of linear systems. In such problems the dependence of the response of the system with time can be stated in a form independent of the configuration of the system. This effectively reduces the number of variables by one.

Another example concerns the flexure of elastic plates in which two opposite sides are simply supported. Practically any condition of supports on the other two sides may be treated, including elastic supports on beams or continuity over flexible beams, etc. If the edges at $x = 0$ and at $x = a$ are simply supported, the deflection function for the plate may be assumed as a trigonometric series in x, with each term multiplied by a function of y. *The partial differential equation governing the deflections of the plate becomes then an ordinary differential equation in* y. Each of the functions of y can be evaluated separately; this evaluation can be made by numerical means even when the formal procedures become too complicated for success.

Special Cases. Under certain conditions this same technique can be used for buckling and for vibration problems. However, the method is not universally useful since many of the most important practical problems do not meet the conditions which are required for these techniques to apply.

In certain types of axially symmetrical problems where the loading or constraint varies with regard to the angle of rotation about the axis in a regular manner, for example as $\sin 2\alpha$, a reduction in the number of variables from 3 to 2 may also be effected. However, at circumferential boundaries in such problems, it may turn out that unwanted residual forces appear. This is the case in certain plate problems. However, a correction may often be made to account for the three-dimensional conditions at or near the boundary where the difficulty occurs.

Characteristic-Value Problems. Many problems which are of practical importance, such as problems of instability, natural frequency of vibration, and others have the property that other than trivial solutions exist only for specific values of some parameter. Since this parameter affects the coefficients generally in the equations which gov-

ern the action of the structure, special techniques must be set up for these problems. The values of the parameters for which solutions exist are called *"characteristic" values or "eigen" values,* and the configurations that correspond to the parameters are called *"characteristic" functions or "eigen" functions.*

Example of Buckling of Bar of Variable Cross Section. As a specific example of such a problem consider the buckling of a simply supported bar of variable cross section. Either the iterative procedure or the step-by-step procedure may be applied. By the *step-by-step method* an assumption is made for the characteristic value, and the step-by-step processes are carried through to observe whether the boundary conditions that are required to be consistent with this assumption are those which correspond with the actual conditions in the problem. If they do not, a new assumption is made and the process repeated. An interpolation or extrapolation from the values obtained in the stages of the analysis leads to a better result each time.

With *iterative procedures,* an assumption is made regarding the configuration of the system and values of the characteristic function required to sustain this configuration are computed. These values will in general differ for the different joints in the structure. If they are nearly alike, the assumed configuration is nearly correct. In most problems of this type the procedure by which better guesses may be made is fairly clear. An outline of the methods applicable to the problem of buckling of bars is contained in one of the author's papers.[3] In some problems the combination of the two procedures is necessary or desirable. This is often the case when relaxation procedures are used for buckling or vibration problems.

Generalization. In general for mixed problems in which ordinary loads and constraints are acting on the structure, the magnitude of the quantity that corresponds to the characteristic value is known and the problem is solved in the usual way. *Such problems are generally nonlinear but the nonlinearity extends only to the effect of the characteristic value, and most of the methods that are available for treating linear problems are applicable to these problems.*

In passing, it might be mentioned that many ordinary problems which are generally considered to be of the usual type have certain properties that are similar to characteristic-value problems under particular conditions. For example, the problem of deflection of a bar on elastic supports leads to a characteristic-value problem when the bar is subjected to no external load, if the spring constants for the

supports can become negative. Under these conditions, a particular value of the spring constant can correspond to a deflected position of the structure, although the position which gives zero deflection is also a position of equilibrium.

Iterative Methods

Successive Approximations. The iterative procedures for numerical analysis may perhaps be most clearly explained with reference to an example. Consider the structure shown in Fig. 9–2, which represents a system of strings a unit distance apart in both directions, crossing each other at right angles. The horizontal component of string tension has a value of unity, the strings are connected together at their junctions, and the deflections of the strings around the boundary are assumed to be zero. It is desired to find the deflections of the nodes or joints where the strings cross each other, for a load of magnitude $P = 1120$ at the corner intersection a. In Fig. 9–3 is shown an element of the string removed for examination. Since the string tension must act in the direction of the string, *the vertical component of the string tension in any panel is equal to the relative deflection of the string in that panel,* as indicated on the diagram.

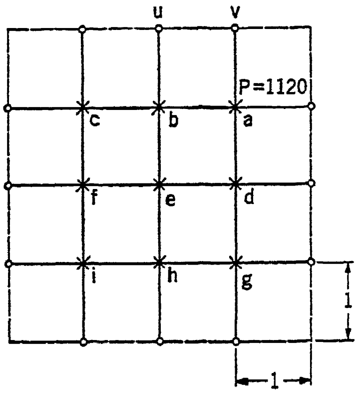


Fig. 9–2. Network of Strings.

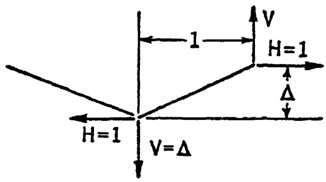


Fig. 9–3. Element of String.

For the time being, we shall consider this problem for its own merit and not as a representation of some other problem. Actually this network of strings can represent reasonably well the conditions in a membrane having a constant surface tension, provided that the deflections considered are relatively small. However, the assumptions that we have made in setting up this particular problem do not limit us to the consideration of small deflections. We have assumed that the strings remain a unit distance apart even though they deflect, and it is further assumed that each joint or node remains in the same plane as the other joints on that particular string.

Numerical Calculations for Crossing Strings. The calculations for this problem are so simple that they will not be given in detail. However, the method used is described in order to permit comparisons of the philosophy and techniques involved with the different procedures that may be used for this problem.

The calculation is carried out in the following way:

(*a*) Initial values of deflections are assumed at all of the joints, *a*, *b*, *c*, *d*, etc.

(*b*) New values of the deflection at each joint are computed from the load on that joint and the deflection of the four surrounding nodes on the strings crossing at that joint.

This calculation is made from the equilibrium condition which can be derived from Fig. 9–3, as follows: The load on any point is the sum of the shears in four strings meeting at that joint and this is, in turn, equal to *the difference between 4 times the deflections at the joint considered and the sum of the deflections at the four surrounding joints.* In other words, where w_n represents the deflection at a point such as n, this relationship is expressed as follows:

$$P_e = 4w_e - w_b - w_f - w_h - w_d \tag{9–1}$$

In this equation P_e represents the load at point e. At each point the values of the loads are known; consequently, equation (9–1) can be solved for the corrected deflection of the joint in terms of the four surrounding deflections and the load at that joint. The procedure is straightforward and, in this problem, will eventually converge to an answer which represents the final equilibrium condition of the structure.

Liebmann Procedure. This procedure may be speeded up considerably if one uses the newly obtained values in each calculation as rapidly as these new values are obtained. This is the so-called Liebmann procedure.[4]

The advantages of this procedure are:

(*a*) That the work can be set up systematically, and it can be carried out by a computer or by an automatic calculating machine.

(*b*) No judgment is necessary after the problem is set up and the solution is started.

(*c*) Errors in intermediate steps are corrected as the calculation proceeds providing that the errors are not in the relationships that are used between the loads and the deflections at the various points.

The disadvantages of this procedure are the slow rate of convergence in many cases and the possible lack of convergence.

The process is applied in about the same fashion when the relationships involved are not linear. This would be the case if, instead of having a constant horizontal tension in the string, the actual string tension remained at a unit value. However, the question of convergence in nonlinear problems becomes somewhat more complicated.

Successive Corrections. The successive approximations technique may be used with a slight modification in which one keeps track only of the changes in deflection at each point. This procedure requires the assumption of a set of original values, and the subsequent computation of the increment in deflection at each point from the preceding set of values. With this technique the order of calculation of successive steps must be fixed in order not to lose track of any of the increments; in general, the procedure can be modified so as to use the latest value at each point.

This procedure has the advantages

(*a*) that it deals with smaller numbers,

(*b*) that extrapolations to a final value can be made from the trend of intermediate results, and

(*c*) that the process can be fairly easily systematized for calculation by relatively inexperienced computers or by means of automatic computing machines.

The disadvantages of this procedure are that the errors in any one step affect all subsequent calculations and produce errors in the final results.

Many of the relaxation or distribution procedures that are commonly used follow along either one or the other of the two methods that have just been described. In general, the true relaxation or physical analogy methods involve only a difference in concept rather than a difference in method from the more abstract mathematical procedures.

Convergence. In nonlinear problems the method of *successive corrections* cannot generally be used, since increments in deflection are not proportional to increments in load in such problems. However, the method of *successive approximations* can almost always be used, even in nonlinear cases. It is obvious that each successive approximation must produce a result nearer the final answer in order for this method to be usable. Otherwise, modifications must be made in the procedure in order to obtain a convergent result.

Divergence. When the sequence of successive approximate values does not converge to a limit the divergence may be regular and of such a nature that a better value can be derived. This occurs in particular when an oscillatory divergence is obtained. This condition is reached,

for example, in such problems as the determination of deflection of a bar subjected to lateral loads and to axial tensions when the axial tension is numerically larger than the critical compressive axial load for buckling of the bar. A similar condition is found for the deflections of a bar on flexible supports when the spring constant for the supports is numerically larger than the "critical" negative spring constant that corresponds to the smallest characteristic value for the structure when no lateral loads are acting on it.

These cases of oscillatory divergence are fairly easy to handle, since it is possible either by inspection or by a formal procedure to determine *the point about which the oscillation takes place.* In some divergent cases, it is even possible to extrapolate backward to the value from which the divergence proceeds.

The relationships between these phenomena are much the same as the relations between the various methods, such as Cesaro's and Holder's, of summing a series which is not convergent. In other words, methods of successive approximation can be successfully employed even when the sequence of calculations does not converge, provided that the series of approximations is summable.

Rapid Convergence to a Practical Answer. Iteration procedures in most cases have a tremendous advantage over other methods when a quick preliminary estimate must be made. In many methods, as for example in the moment distribution procedure, only a few cycles of calculation are required to obtain results of a practicable degree of accuracy. And in buckling or vibration problems in general only one cycle is needed to get a reasonable answer together with some information regarding the possible errors in such an answer.[3] *However, where extreme precision is required, step-by-step methods may give better and faster results. Their disadvantage in general is in the amount of work necessary to form even a first estimate of the answer.*

Step-by-Step Methods

Approximate *vs.* Exact Methods. Step-by-step methods are generally based on one of the several fairly well-known methods for the step-by-step numerical integration of ordinary differential equations. These methods differ in detail and in their accuracy, but they have a number of characteristics in common. In applications to physical problems we may distinguish between:

(*a*) Methods that are "exact," which generally are methods applicable to discrete joint problems; (*b*) Methods that are approximate,

which generally are methods applicable to continua. However, methods of the former type may be used with problems of the latter and conversely.

Step-by-step methods are most conveniently used when complete boundary conditions are known at some one point as in the case of a dynamic system subjected to an impulse, etc. This point is then a convenient place for starting the solution which progresses by steps and increments to any other point desired. However, *such methods must cover all of the region between the points considered*, and therefore may involve needless calculation if the intervening region is not of interest. The relationships which apply in each step may be explained most clearly with reference to a problem.

Example. If a simple system consisting of a mass supported by a spring is subjected to a transient force beginning at time $t = 0$, we may start the calculation at this time and compute successively, at desired increments of time, the magnitudes of acceleration, velocity, and displacement of the mass. The relations between these quantities may be approximate, as obtained by use of difference-equation analogies; or they may be exact, as obtained by use of steps in an exact solution for the particular conditions under consideration. The accuracy gained with the analytical solution in each step is offset by the complications in the use of such solutions. Hence in many cases the approximate methods, with suitable refinements to insure accuracy, are preferable. An example of the so-called analytical method in the step-by-step procedure is given by Myklestad [5] in a treatment of the vibrations of beams.

Boundary Conditions. *Where incomplete boundary conditions are known at the necessary starting point and the remainder of the boundary conditions are available at some other point, the step-by-step method has to be developed in several parts.* Briefly, the procedure is outlined as follows:

(*a*) Sufficient boundary conditions are assumed at the starting point to enable the calculation to proceed in the normal fashion. The calculations are carried to some terminal point where additional boundary conditions are available. In general, the derived conditions at these points will not check with the given boundary conditions.

(*b*) For linear problems, the influence of changes in the initial boundary conditions are determined by repetition of the calculations in (*a*). It may be convenient to consider only the changes in values rather than to recompute the whole problem for the changed conditions.

(*c*) By linear combinations of the effects of changes in the given boundary conditions the necessary initial conditions are deduced to give the desired terminal conditions. This calculation involves merely the solution of a small number of simultaneous equations in the case of linear problems.

The procedure described is a generalization of the one used by Myklestad [5] for determination of the response to dynamic loads of beams with simple supports or with other end conditions. The procedure has been used by Southwell [6] and others in the determination of critical buckling loads for bars.

Nonlinearity. The solution by step-by-step methods of nonlinear problems is complicated by the fact that the changes in initial conditions do not produce linear changes in the terminal conditions. *Consequently, a complete recalculation must be performed with each change in the initial conditions.* However, the interpolation process may still be used to avoid making the whole work completely a trial-and-error procedure. Particularly when one is close to the final answer a sort of quasi-linear relation may be used.

General Comments. In general, step-by-step methods are most successfully used with *hyperbolic differential equations* or in problems arising from such equations, whereas iteration procedures are most successfully used with *elliptic equations.* However, each procedure is not necessarily limited to a particular type of problem. This is not generally recognized. In applying one or the other of these methods to types of problems to which they are not obviously applicable, precautions must be used in establishing the necessary conditions. An illustration is given, in the next paragraph, of the technique that can be used with the step-by-step method when it is applied to a problem arising out of an elliptic differential equation.

Illustration of Step-by-Step Analysis. The step-by-step method becomes complicated when applied to other than one-dimensional problems. However, the principles of application of the method are always the same. An illustrative example will indicate the technique for a two-dimensional case, and *will also demonstrate that the initial and terminal conditions need not be boundary conditions in the usual sense.*

Consider again the problem shown in Fig. 9–2. The problem is solved by a step-by-step calculation in five operations, as shown in Table 9–1. Only two of these operations involve step-by-step solutions. The entries in the table are completely described. Column (1)

is the first trial sequence. Deflections at points i and e are assumed, and the other deflections are computed by successive steps from equation (9–1). Two different values are determined for point b depending on the way the point is reached. The dual value is of course impossible and is obtained because of inconsistent assumptions of the values at i and e.

Column (2) shows the computation of a correction pattern for a change in point e. Here again different values are obtained for b for the two routes selected. Since this is a linear problem, a multiplication factor for column (2) can be selected to give an equal and opposite inconsistency at b to that obtained in column (1). This is shown in column (3). Then column (4) indicates a consistent solution, which however corresponds to a load at a of twice the desired magnitude. Therefore, the final step to obtain column (5), is merely to divide column (4) by two.

TABLE 9–1. STEP-BY-STEP CALCULATION OF STRING NETWORK OF FIG. 9–1 USING EQUATION (9–1)

Point	Operation using equation (9–1)	(1) Trial Sequence	(2) Correction Pattern	(3) 0.2×(2)	(4) (1)+(3)	(5) 0.5×(4) Final
i	Assumed deflection (1) and (2)	30	0	0	30	15
f, h	Computed deflection from i	60	0	0	60	30
e	Assumed deflection (1) and (2)	120	100	20	140	70
c, g	Computed deflection from f	90	−100	− 20	70	35
b, d	Computed deflection from c	300	−400	− 80	220	110
b, d	Computed deflection from e	180	200	40	220	
b	*Difference in computed deflections*	*120*	*−600*	*−120*	*0*	
a	Computed deflection from b				670	335
a	Load				2240	1120
a	Desired load				1120	1120

PHYSICAL ANALOGY METHODS

Relaxation. The methods of this chapter are explained from the point of view of a structural or mechanical model, although the descriptions are general and the techniques described may be applied to problems of flow, to dynamical problems, or in general to any problem that can be formulated mathematically or physically.

The method of successive relaxation of constraints [7] is based on the concept that *the structure is maintained in a continuous state, but it has acting on it residual loads which are not statically consistent with the*

desired loading. The "residuals" are modified or reduced by introducing arbitrary changes in displacement. There are many ways of keeping track of the calculations; but in general a record must be kept at each joint of both the displacement and the residual force which must be liquidated. The record may be kept in terms of totals of these quantities, or in terms of increments, and part of the record may be implicit or may involve a mental calculation or observation. This is the case in the usual moment distribution procedure, where the residual moment at a joint is the difference in the moments in the members on each side of the joint.

Distribution vs. Relaxation. Although the essential features of the method are not changed by the bookkeeping, it seems worth while to make a distinction between "distribution" procedures and "relaxation." The greater formality in the former methods simplifies the records but permits less latitude and opportunity for judgment in the calculations. The "liquidation" of residuals by relaxation involves the choice of arbitrary patterns of deformation. This is done in most cases by means of a so-called "operations table." The operations table is merely a simple way of keeping track of the effects of patterns of displacement or deflection.

Block Movements. By appropriate combinations of the items in the operations table, group displacements may be easily considered. An equally good way of keeping these data is to record the quantities directly on a diagram and to make necessary group or block corrections by changing all values proportionately whenever it seems expedient to do so. For example, one can assume initial values of the deflections and compute from these the magnitudes of the loads associated with these deflections. Then by means of an appropriate proportionality factor, all of the deflections and loads may be modified simultaneously to make the scale of the deflections more nearly correct. This process is outlined in Fig. 9–4 for the string network problem of Fig. 9–2.

Example. The boxes in Fig. 9–4 represent the joints in Fig. 9–2. In each box both trial deflections and loads and total deflections and loads are recorded. The trial values are "operations" or "group relaxations." *At any stage of the calculations any proportion of any one or several of the trial sets of values may be added to all the totals.* An explanation is shown in the figure of the steps in the calculation. Only three steps were considered, and the results are fairly good as can be seen by comparison with Table 9–1 which gives an exact solution for

the problem. In this illustration the total loads are recorded rather than the residuals since the residual is equal to the total except for a. In general it is more convenient to keep track of the residuals but the changes in the tabulation required to do so are obvious.

	c				b				a				
	Trial		Total		Trial		Total		Trial		Total		
Line	Defl.	Load	Defl.	Load	Defl.	Load	Defl.	Load	Defl.	Load	Defl.	Load	Explanation:
(1)	30	40			60	−30			200	680			Trial pattern
(2)			48	64			96	−48			320	1088	1.6 x (1)
(3)	−20	−85			10	45			10	20			Trial correction
(4)			32	−4			104	−12			328	1104	0.8 x (3) + (2)
(5)	3	−1			10	17			10	20			Trial correction
(6)			33.5	−4.5			109	−3.5			333	1114	0.5 x (5) + (4)
	f				e				d=b				
(1)	20	0			40	0							
(2)			32	0			64	0					
(3)	−5	−2			5	10							
(4)			28	−1.6			68	8					
(5)	3	−2			10	14							
(6)			29.5	−2.6			73	15					
	i				h=f				g=c				
(1)	10	0											
(2)			16	0									
(3)	−3	−2											
(4)			13.6	−1.6									
(5)	1	−2											
(6)			14.1	−2.6									

FIG. 9–4. RELAXATION SOLUTION FOR A STRING NETWORK.

Distribution. Distribution procedures are somewhat more formal than most relaxation methods, but generally require less judgment on the part of the computer. This may not always be an advantage but it does make possible the employment of relatively unskilled computers who can follow a routine and who do not have to use judgment in selecting types of block operations.

It is assumed that the reader is familiar with the ordinary moment distribution method.[8] The method may be generalized, with the provision that the terms *members, forces, etc.*, may not apply to specific physical quantities. The terms are used for convenience in the description.

(*a*) The residual force at a joint is computed from the forces at the ends of the "members" meeting at the joint.

(*b*) This residual is "distributed" among the members meeting at the joint in proportion to their "stiffness" — the other joints are kept "fixed" in this process.

(*c*) The forces at the other ends of the members are calculated from "carry-over" factors applied to the distributed forces.

Example. In the case of the string network of Fig. 9–2, the stiffness of all members meeting at a joint is the same. Therefore one-fourth of the residual is distributed to each string, as a shear in the string. If the sign convention is adopted that a positive shear exerts an upward force on the joint, the sum of all the shears meeting at a joint must equal the load at the joint, and the carry-over factor of each string is -1. With these constants, the usual moment distribution techniques apply to the analysis.

A somewhat more complicated definition of stiffness and carry-over factor is required to analyse the structure shown in Fig. 9–1, but the procedure is entirely feasible.

Continuity Restoration or Compatibility. The writer's compatibility scheme is related in some respects to the orthodox method of cutting sufficient gaps in a structure to render it statically determinate; *then to close the gaps by statically self-balanced systems of loading.* However, the essential difference is in the way the gaps are chosen, and in the manner of closing them. *In this method, statical equilibrium is always maintained but internal discontinuities or departures from compatibility are permitted, and gradually eliminated.* For the actual computation, schemes similar to relaxation or distribution may be used, of course.

Consider again the problem of Fig. 9–2, and adopt the convention that an arrow along a string points along an upward slope of the string and has a magnitude equal to the shear, proportional to the difference in deflection. Then we may consider the flow of shear in the strings and derive two theorems, analogous to Kirchoff's Laws:

(*a*) If downward loads at a joint are considered as inflows, the total of all the flows into a joint must equal the total flow out of the joint.

(*b*) Since the differences in deflection are proportional to the shear flows, the sum of the flows in any closed circuit must be zero. This is the condition of compatibility for the strings, and means that the sum of the differences in elevation in any closed path must be zero.

Discontinuities. We can determine shears, in all the strings, that are statically consistent with the loads by starting at any joint and successively guessing at values where necessary, to satisfy condition (*a*). A set of values so assumed is shown in Fig. 9–5(*a*), to the upper left of the diagonal line. These flows do not satisfy condition (*b*), which means that we have discontinuities in the network; but we do not proceed to determine them. We may put in any constant flow around any closed circuit in which we wish to eliminate the "incompatibility." The structure may be brought into a continuous state by this means,

although a more systematic approach may be desirable. Such an approach is shown in Fig. 9–5.

Restoring Continuity. In each elementary unit or block the lack of compatibility is indicated above the short horizontal line. For example in the third block from the upper left hand corner block *abuv* the clockwise flow is 260, the counterclockwise flow is 400, and the total correction flow to balance these is the difference between them, or 140 clockwise. Actually this total is provided by shear flows in only three strings since the boundary of the network is at constant elevation. The required flow in each string would be 46.67. However, this flow would change the conditions in adjacent units and require additional corrections in them, except for the units on the diagonal where conditions of symmetry keep the flows always balanced.

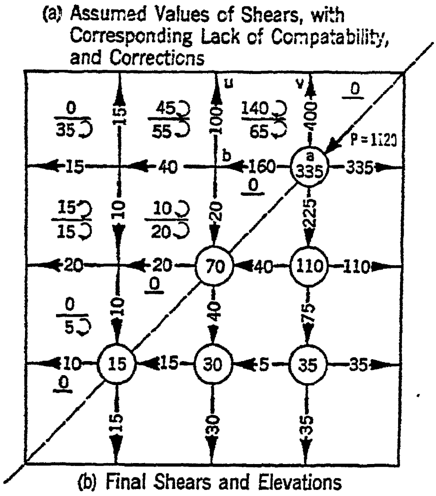


FIG. 9–5. ELEVATION OF NODES IN STRING NETWORK.

The correction flows in each unit which are consistent with the original unbalance, as well as the additional unbalance caused by adjacent flows, are shown below the short horizontal line in each block. These flows are the flows for each string. These correction flows *combined with the original guesses* give the resultant or final shear flows which are indicated on the lower right, in Fig. 9–5(*b*). The elevations of the network can be readily computed from the flows.

A more detailed account of this procedure applied to the torsion problem is presented in a thesis written under the writer's direction by Mr. E. C. Colin.[9] The compatibility procedure has also been applied to plane problems in elasticity, by use of framework analogies. These procedures are described in the next chapter.

PLANE STRESS PROBLEMS

Framework Analogies and Other Methods. Several numerical methods of solution of plane problems in elasticity have been described.[10–16] Both framework analogies and finite difference nets have been used. One of the earliest solutions is that due to Richard-

son[10] who used a finite difference representation of Airy's stress function. This stress function satisfies the same equations as the deflections of a thin elastic plate; an analogy can be made between ordinates of the stress function and deflections of the plate, and between errors in compatibility in the stress problem and the load on the plate. *It is interesting to note that with this analogy a relaxation procedure for the plate problem is equivalent to a compatibility procedure for the plane elasticity problem.* Recently, a relaxation technique for Airy's stress function was described by Zienkiewicz.[11] Another early solution of plane stress problems was given by Ferris[12] who used a finite difference representation for the equations governing the displacements.

Lattice Analogies. Framework or lattice analogies have been described by McHenry[13] and Hrennikoff.[14] These analogies consider the solid material replaced by a system of pin-connected bars, the areas of which are so chosen as to represent the physical behavior of the solid as closely as possible. Lattices of the types considered are shown in Figs. 9–6 to 9–9. With these, the effective widths of the bars in the hexagonal lattice are equal to the spacing of parallel bars; and for the square lattice, three-fourths of the lengths of the bars between points where bars cross each other. Boundary bars have one-half these widths. *With these areas, the value of Poisson's ratio turns out to be $\frac{1}{3}$. Unless a special technique is adopted, this is the only value that can be used with the lattice.*

Hrennikoff uses essentially a distribution procedure, although the relaxation procedure may be equally effective. McHenry derives an equation for the displacements of any point in terms of the displacements of surrounding points, and adopts an iteration process to compute the displacements.

The application of a compatibility procedure to the hexagonal lattice has been studied by Austin[15] in a thesis under the writer's direction. Studies of both square and hexagonal lattices with compatibility and with displacement procedures have been made by Dauphin.[16]

Compatibility Procedure for Hexagonal Lattice. To illustrate further the applicability of the compatibility procedure a sketch of the way it can be applied to the hexagonal lattice is indicated. There are two conditions which must be fulfilled by the stresses and strains in the bars:

(*a*) Each joint must be in statical equilibrium.

(*b*) Each hexagonal unit of the type shown in Fig. 9–6 must have

compatible strains in the bars; that is, the deformations must be such that bar bc, for example, can fit into the space allotted for it by the deformations of the other bars.

Designating the force in any bar by the letters denoting the ends of the bar, and adopting the assumption of elasticity, or of proportionality of stress and strain, we can derive the following conditions for the bar forces:

Statics, at each joint

$$oa - od = oc - of = oe - ob \tag{9-2}$$

Compatibility, in each hexagon

$$oa + ob + oc + od + oe + of = ab + bc + cd + de + ef + fa \tag{9-3}$$

A slight modification must be made in equation (9–3) for bars at a boundary; the corresponding term must have a coefficient of 2.

Adjusting Hexagonal Units. In Fig. 9–6 a statically self-balanced system of forces in the bars is shown. The effect of this system on the compatibility is to change the error in compatibility by 12 for an interior hexagonal unit. Considering the system in Fig. 9–6 as a positive unit in the sense shown, and the compatibility change in the figure as positive, the effects on adjacent hexagonal elements of a positive unit change in any one element is shown in Fig. 9–7.

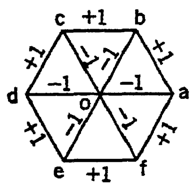


Fig. 9–6. Basic Hexagonal Unit.

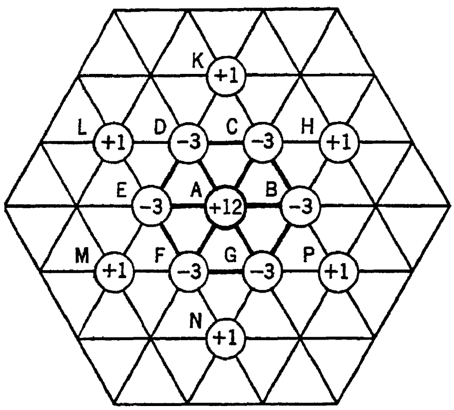


Fig. 9–7. Effect on Adjacent Units of Correction in Unit A.

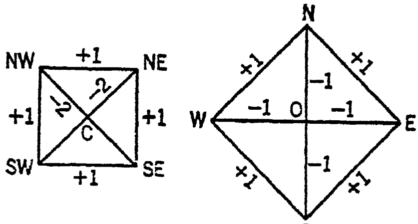
The procedure for the analysis is then as follows:

(*a*) Assign statically consistent forces to the bars, starting with the applied loads.

(*b*) Compute the error in compatibility in each hexagonal element.

(*c*) Correct the elements by adding self-balanced changes in forces, taking into account the disturbances created in adjacent elements.

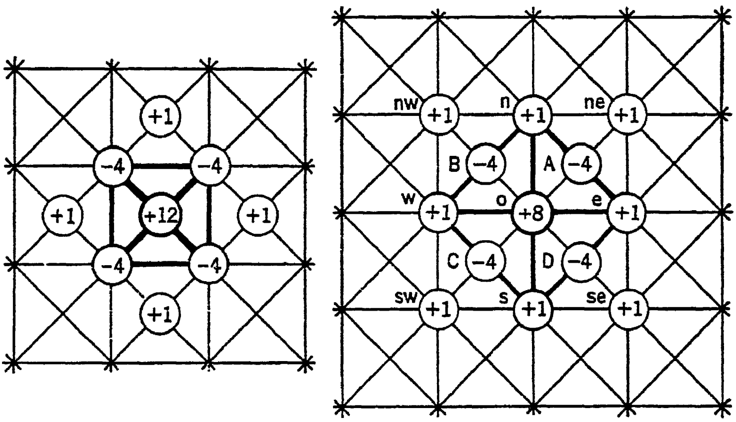
The details of the procedure are subjected to variation in the same way as for any of the other methods of analysis. Two additional items must be noted: First, compatibility of all elements insures internal and external continuity only if there are no holes in the body and no external constraints; otherwise, a subsidiary correction must be made. Second, one can hasten convergence of the procedure by considering large units and small ones in turn.



(a) Diagonal Unit (b) Normal Unit

FIG. 9–8. BASIC UNITS FOR SQUARE LATTICE.

Square Lattice. The compatibility procedure for the square lattice is somewhat more complicated than for the hexagonal lattice. Two types of elements are required: a diagonal unit in which the interior members are diagonal bars, as shown in Fig. 9–8(*a*), and a normal unit, as in Fig. 9–8(*b*). This figure also shows statically self-balanced systems of bar forces in the fundamental elements. With these correction systems, the effects on the compatibility of adjacent elements are shown in Fig. 9–9, where the effect is



(a) Diagonal Unit (b) Normal Unit

FIG. 9–9. EFFECT ON ADJACENT UNITS OF CORRECTIONS IN DIAGONAL AND NORMAL UNITS.

given by the number in the circle at the center of each element. The compatibility condition for any element is as follows:

$$\begin{bmatrix}\text{Sum of forces in}\\ \text{peripheral bars}\end{bmatrix} = \begin{bmatrix}\text{Sum of forces in all}\\ \text{four radial bars}\end{bmatrix} \qquad (9\text{–}4)$$

In the diagonal unit, each diagonal is therefore counted twice since each half is a radial bar. The analysis is carried out in the same way as for the hexagonal lattice.

The stresses in the interior are determined from the bar forces in the following way: Consider vertical and horizontal planes passed through a joint as shown in Fig. 9–10. The planes divide the vertical and

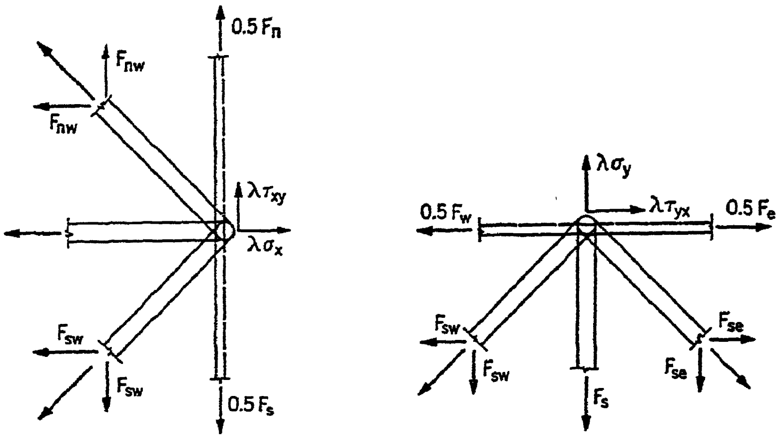


FIG. 9–10. DETERMINATION OF STRESS AT A NODE FROM FORCES IN LATTICE BARS.

horizontal bars in two, and accordingly give a system of bar forces as shown in the figure. The normal and shearing stresses in the body at the joint are determined from the equilibrium conditions of the force systems shown, where λ is the length of the vertical and horizontal bars, σ is the normal stress and τ the shearing stress.

With this interpretation of stress it can be shown that under all conditions the usual relation holds:

$$\tau_{xy} = \tau_{yx} \tag{9–5}$$

COMPUTATION OF INFLUENCE COEFFICIENTS

Fundamental Principles. In the calculation of deflections, stresses, moments, etc., in structures by use of difference equations or by other numerical methods, a particular effect often is desired for a unit load applied in turn at all the joints or nodes of the network into which the structure is divided. In many cases it is inconvenient because of the amount of work involved to apply unit loads successively at all the desired points. Under certain conditions this calculation can be made directly, and an influence surface, line, or volume can be obtained.[17]

An influence surface can be computed when the particular effect desired is a linear function of certain of the deflections at particular joints in the structure, and when the relationship between deflections and loads for the structure is linear. When these conditions apply, the influence of a unit load at any point m is obtained as a deflection at point m due to loads applied to the structure in accordance with the following theorem:

Theorem. "Where a particular effect Q is a linear function of the deflections $w_a, w_b \cdots w_k$ at points $a, b, \cdots k$, respectively, as in the equation

$$Q = Aw_a + Bw_b + \cdots Kw_k$$

and if deflections and loads are linearly related, the influence on Q of a unit load at any point m is obtained as the deflection at the point m due to loads of magnitude $A, B, \cdots K$, applied at points $a, b, \cdots k$ respectively." It may be noted that the application of the procedure is not limited to problems in which difference equations are employed.

Illustrations of Influence Coefficients. If, for example, the influence is desired for moment in the x-direction at a point in a plate, such as at c in Fig. 9–1, we may obtain the influence coefficients by determining the deflection of the plate due to loads applied in the following way.

The moment desired is formulated in terms of deflections by the equation:

$$M_x^c = \frac{1}{\lambda^2} \cdot \frac{EI}{1 - \mu^2} [2(1 + \mu)z_c - z_e - z_w - \mu z_n - \mu z_s] \qquad (9\text{–}6)$$

Therefore, the loads to be applied are:

at c

$$\frac{2(1 + \mu)EI}{\lambda^2(1 - \mu^2)} \qquad (9\text{–}7)$$

at e and w

$$-\frac{EI}{\lambda^2(1 - \mu^2)} \qquad (9\text{–}8)$$

and at n and s

$$-\frac{\mu EI}{\lambda^2(1 - \mu^2)} \qquad (9\text{–}9)$$

The deflections due to these loads will be the influence coefficients desired.

Müller-Breslau Principle. As another example, if in Fig. 9–2, the influence is desired for the deflection at point a of deflections at u, v,

and w on the boundary, these cannot be obtained directly from the theorem. However, from the Maxwell-Mohr-Müller-Breslau principle of influence lines, these influences are obtained as the reactions at u, v, and w due to a unit load at a. Since these reactions are the vertical shears at these points, they may be obtained from Fig. 9–5. The results are as follows:

Influence at a of unit deflection at		
u	v	w
$\frac{335}{1120}$	$\frac{110}{1120}$	$\frac{35}{1120}$

Increasing Accuracy

Mistakes and Checks. One of the major difficulties in the use of numerical procedures arises from numerical errors or mistakes in the calculations. *It has already been pointed out that successive approximation methods are most satisfactory from the point of view of automatically correcting mistakes unless the same mistakes are repeated.* In any case, some system of checks on the accuracy of the result is desirable. Sometimes overall checks can be made very simply; for example, in Fig. 9–5(*b*), the total shears around the boundary for one-half the structure should add up to one-half the total load. Similar checks can be made in other problems. Usually the checks are more obvious in methods that are derived from physical analogies.

With any method, it is desirable to compute and study the residual forces or residual errors in compatibility, from the final computed values of stresses or deflections. Then if a mistake has been made it is not necessary to locate the source. One has merely to liquidate the residual — in this case, the mistake — to get the result desired.

Finer Nets. Aside from mistakes, other sources of error are in the representation of continuum problems by discrete joint problems. One way of making errors smaller due to this cause is to use a more finely divided net. However, the increase in effort required is not often commensurate with the increase in accuracy.

For example, in the numerical calculation of the buckling load of a bar of constant section with hinged ends by use of finite differences, the errors in the coefficient for critical load are as follows:

n = No. of nodes in length of bar	Error (exact value = 9.87)	% Error
2	1.870	18.9
3	0.870	8.8
4	0.498	5.0
10	0.080	0.81
20	0.020	0.20
90	0.001	0.01

Similar results are found in other problems. However, in two-dimensional problems, the amount of work increases much more rapidly than the number of points, but the increase in accuracy may lag far behind the increase in effort.

Net Variation. In some cases, where quantities vary rapidly, a small net spacing is needed. But this may be required only in part of the structure, and it is wasteful of time and accuracy to use the small net everywhere. In such circumstances it is often expedient to use *two different sizes of mesh,* with the fine spacing where it is necessary, the coarse spacing elsewhere.

Extrapolation. It was pointed out by Richardson [10] that more accurate results could be obtained from difference equation solutions by *extrapolating from the results of analyses with several different sizes of net.* This matter has been studied further by Salvadori [18] who indicates that a series expression for the error may be formulated and extrapolations based on this formulation. He points out, however, that an extrapolation from only two trials may not be safe, and suggests a minimum of three calculations with different net sizes. This is of course inconvenient. In the buckling problem for which the errors are cited it can be shown that the first term in the series of errors depends on $1/n^2$, and the second term probably on $1/n^4$. An extrapolation using only the first term gives an error of 0.070 based on $n = 2, 3$ calculations, and an error of 0.020 from $n = 3, 4$ calculations; whereas an extrapolation from the $n = 2, 3, 4$ calculations using two terms in the error series gives an error of less than 0.001.

The formulation of the error in replacing a derivative by a difference generally involves errors which are expressible as a series in which the first term involves the square of the net spacing. Ordinarily, only even powers of the spacing are involved in the additional terms, but others may also be involved if the boundary conditions have to be approximated in different ways from the schemes used in the interior.

Higher Orders. If higher order approximations of derivatives by differences are used, the power of the first term in the formulation of the error may be increased to 4, but a value of 2 or 3 may appear if boundary conditions are not carefully treated. With second order approximations used in the buckling problem cited, the errors are materially reduced, as follows:

n = No. of nodes in length of bar	Error	% Error
2	0.270	2.74
3	0.051	0.52
4	0.016	0.16
8	0.001	0.01

Almost as much improvement can be obtained in other problems as well. The extrapolation procedure can be used here also; and with the first term involving $1/n^4$, the extrapolation from the $n = 2, 3$ calculations gives almost exact results.

Other Methods. Other methods of obtaining increased accuracy are also based on the relations between finite differences and derivatives. For example, the second derivative of a function can be expressed in terms of central differences, as follows:

$$\lambda^2 \frac{d^2f}{dx^2} = \delta'' - \frac{1}{12}\delta^{\text{iv}} + \frac{1}{90}\delta^{\text{vi}} - \cdots \tag{9–10}$$

With only the first term on the right of equation (9–10), *the errors in the integration of an equation are proportional in general to the square of the mesh size,* λ; with two terms, the errors are proportional to the fourth power; etc., as has been pointed out in the foregoing paragraphs. However, the form of the difference equations that are obtained with additional terms in equations such as (9–10), is much more complicated. For this reason, and because of difficulty in the accurate representation of boundary conditions with higher order differences, other methods have been proposed.

Modifying Terms or Loads. A method that is useful in ordinary differential equations is to modify the terms on the right-hand side of a relation between differences and derivatives, corresponding to the inverse of equations such as (9–10). For example,

$$\delta'' = \lambda^2 f'' + \tfrac{1}{12}\lambda^4 f^{\text{iv}} + \tfrac{1}{360}\lambda^6 f^{\text{vi}} + \cdots \tag{9–11}$$

If the quantity on the right can be computed, the integration of the differential equation is replaced by a summation of a difference equa-

tion. This procedure, for the second derivatives described here, corresponds to modifying loads on a beam for the purpose of computing the moments in the beam accurately at a series of points. The procedure has been described by Southwell[6] in terms of the mathematical aspects, and by Newmark[3] in terms of load modifications on a beam. The latter representation has certain advantages when discontinuous functions must be dealt with, and also permits rapid approximate estimates to be made by use of one's judgment.

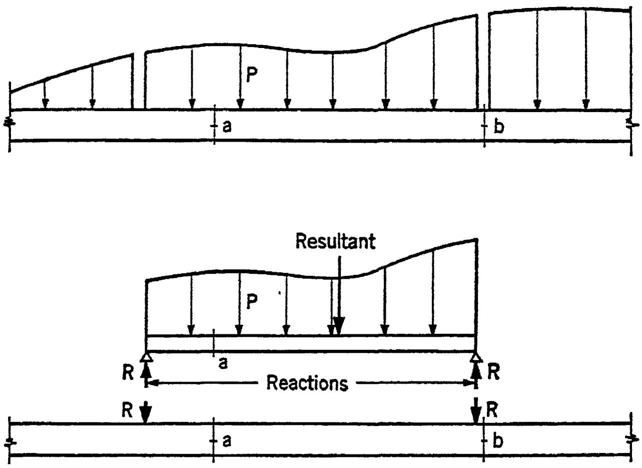


FIG. 9–11. REPLACEMENT OF DISTRIBUTED LOAD WITH STATICALLY EQUIVALENT CONCENTRATIONS.

Example of Modified Beam Loads. The basis of the physical approach is indicated in Fig. 9–11, which shows a beam subjected to a variable distributed load. That part of the load diagram represented on the figure by P has a statical equivalent equal to its resultant. However, it is not convenient to consider resultants of the various parts of the load diagram with their variable spacing. Since the resultant is also statically equivalent to the reactions on a simple beam of that part of the load diagram considered, the recommended replacement is by the sub-stringer reactions R as shown on the figure.

With this replacement, which is applicable to concentrated or discontinuous loadings as well, the correct moments and shears are obtained at all points outside of the region covered by P, and in particular at points such as b at the ends of the interval. To find the moments and shears at a within the interval, it is necessary merely to add the sub-stringer moments and shears at a to those obtained from the replacement forces R acting on the main beam.

Generalization. This procedure can be extended to other problems, but in general, it is useful only for problems with *a single independent variable.* For other problems, it is desirable to use a procedure recently described by Fox.[19] In this procedure calculations are carried out for the usual finite difference representations using only the first term in equations such as (9–10) and (9–11). Then residuals are computed from the more accurate representation by means of equations such as (9–10). These residuals give rise to a correction solution, which is in turn corrected, and the process continued until the corrections are negligible. Results of surprising accuracy are obtained by this process with only a relatively small number of mesh points.

Combined Numerical-Analytical Procedures. Possibly the simplest way of obtaining an increased accuracy with numerical procedures in continuum problems is to use the numerical procedure for the purpose of computing a correction to results obtained by methods directly applicable to solid bodies, etc. For example, in plane problems in elasticity, an approximate result may be obtained from formulas derived by methods of ordinary mechanics. These results can be expressed in terms of a stress function which usually will not satisfy the conditions of compatibility. The errors in compatibility may then be computed, and by means of the stress function — plate analogy, the correction stresses may be determined to account for the errors in compatibility.

This is a well-known analytical procedure, but it may be used also when the correction terms are determined by numerical processes. When the original assumptions are fairly reasonable, the corrections are relatively small, and need not be computed numerically with a high degree of accuracy in order to give a combined result that is as precise as necessary.

Torsion Example. Another illustration of this technique may help to show its generality. Consider the problem of torsion of a relatively compact cross section of arbitrary shape. The stress function for torsion must satisfy the equation:

$$\nabla^2\phi = -2G\theta \tag{9–12}$$

where G is the shearing modulus of elasticity, and θ the relative angle of twist. The boundary conditions are:

$$\phi_{\text{bdry}} = 0 \tag{9–13}$$

An approximation to this function can be made by the relation

$$\phi_0 = \frac{G\theta}{2}(a^2 - x^2 - y^2) \tag{9–14}$$

where a^2 is arbitrarily chosen to make the values of ϕ_0 as small as possible at the boundary. Now we introduce a correction function ϕ_c such that

$$\phi = \phi_0 + \phi_c \tag{9–15}$$

The function ϕ_c must satisfy the equation

$$\nabla^2\phi_c = 0 \tag{9–16}$$

and the boundary values are equal and opposite to the boundary values of ϕ_0.

By good judgment in the choice of a^2, the corrections will be fairly small. They can be computed by finite difference methods and the complete solution obtained as a combination of the results of an analytical solution ϕ_0 and a numerical set of values of ϕ_c. Other functions may be used in equation (9–14) for better results in sections of unusual shape. Obviously this whole technique may be extended to other types of problems, and *even to nonlinear problems,* and it combines advantages of both formal analytical procedures and numerical methods.

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CHAPTER 10

A SURVEY OF THE APPROXIMATE SOLUTION OF TWO-DIMENSIONAL PHYSICAL PROBLEMS BY VARIATIONAL METHODS AND FINITE DIFFERENCE PROCEDURES

THOMAS J. HIGGINS *

Introduction. Variational methods and finite difference procedures afford powerful means of obtaining approximate mathematical solutions of two-dimensional boundary value problems in engineering and physics which are intractable to exact analysis. Although a solution by variational methods requires a mathematical technique very different from the numerical analysis used for a solution by finite difference procedures, the two means are yet complimentary. For often it happens that a most feasible means of solving a given boundary value problem is to cast it as a variational problem, approximate the mathematical functions involved by corresponding difference expressions, and then solve the resulting problem in finite differences by a suitable finite difference procedure. Accordingly, a knowledge of both variational methods and finite difference procedures is useful to the worker in numerical methods. Yet, despite this significance, no considerable exposition of these approximate methods is to be found in standard reference texts on applied mathematics or in those covering a given domain of technical or physical theory; nor, for that matter, is other than limited mention made therein of sources, original or secondary, that contain such exposition.

The purpose of this presentation is to fill this gap by furnishing an integrated survey of the literature essential to study by one desiring to gain knowledge of both the general theory of these methods and the actual technique of using this theory. It is the writer's experience that after study of the general theory, facility of use is quickest gained by careful study of all available literature devoted to solution of one of the classical problems of applied mathematics. Technique is illus-

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trated here by account of the application of these approximate methods to St. Venant's torsion problem.* Elsewhere [1,2] the reader can find account of the application of some of these methods to the eigenvalue problems of the vibrating membrane and the wave guide of uniform cross section.

List of Major Symbols

(Arranged approximately in order of introduction into the text.)

μ: Modulus of rigidity
τ: Angle of twist per unit of length
M: Torsional moment
C: Torsional rigidity
ϕ: Torsion function
ψ: Conjugate torsion function
Ψ: Torsional stress function
S: Area of cross section of prism
s: Boundary curve of S
$U(\Psi)$: A parameter defined by the integral equation (10 − 2)
$I(\Psi)$: A parameter defined by the integral equation (10 − 3)
$J(\Psi)$: A parameter defined by a certain integral equation not given in text
U: The minimum value of $U(\Psi)$; and the stored elastic energy per unit of length
I: The minimum value of $I(\Psi)$
J: The maximum value of $J(\Psi)$
T_m: The maximum shearing stress
ϵ: An error function
n: An integer subscript indicating an n^{th} approximation

* For accounts of the general mathematical theory see: S. Timoshenko, *Theory of Elasticity* (McGraw-Hill, 1934); I. S. Sokolnikoff, *Mathematical Theory of Elasticity* (McGraw-Hill, 1946); A. E. H. Love, *The Mathematical Theory of Elasticity* (4th ed.; University Press, Cambridge, 1934).

The notation and terminology of this paper are those of Love: μ denotes the modulus of rigidity; τ the angle of twist per unit length; and the other symbols are defined as introduced. To avoid redundance, symbols are used in the text in lieu of the corresponding phrase. Again, inasmuch as determination of either the torsional moment M or the torsional rigidity C is tantamount to determination of the other, for ($M = \tau C$), mention is made only of M — though in some of the papers discussed it is the expression for C, and not M, that is given.

For an interesting sketch of Saint-Venant's life and work see: K. Pearson, "M. Barre de Saint-Venant," Nature **33**, 319–321 (1885–1886).

[1] The bibliography with references numbered 1–140 begins on page 192.

∇^2: The mathematical operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

grad^2: The mathematical operator $\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$

Variational Methods

Ritz's Method. In 1908 the Swiss physicist Walther Ritz published a memoir [3] wherein he propounds a new method of solving boundary value problems; and in this and subsequent memoirs [4, 5] he effects by this method approximate solutions of problems which had long defied the efforts of analysts. Among the classical problems treated in the first memoir is the Dirichlet problem for a "singly-connected" region. Inasmuch as the torsion problem for a "singly-connected" prism * is interpretable as a first Dirichlet problem,** the solution for a particular cross section is afforded by Ritz's method — sometimes termed the Ritz-Rayleigh method, Lord Rayleigh [6] having pointed out that Ritz's procedure has much in common with approximate methods used earlier by him.

Application of Ritz's method to a boundary value problem hinges on expressing the problem as one in the calculus of variations, *a transformation usually effected with ease inasmuch as the definitive partial differential equations of most physical problems can be formulated from one of the well-known minimum or maximum "principles" of physics — for example, Hamilton's, or Maupertuis', or Castigliano's principle.* Transformed, the problem reduces to determination of a function which minimizes a certain integral expression subject to specified boundary conditions. The torsion problem, for example, can be interpreted as the search for the stress function

$$\Psi \equiv [\psi - (x^2 + y^2)/2] \qquad (10\text{–}1)$$

that minimizes the integral expression

$$U(\Psi) = \left(\frac{\mu\tau^2}{2}\right)\int_S \text{grad}^2\,\Psi\,dS \qquad (10\text{–}2)$$

* The torsion problem for a prism of multiply-connected cross section is also solvable by Ritz's method; however, the boundary conditions are much more complicated than for the singly-connected prism. For an inclusive discussion of the formulation of the boundary value problem of the multiply-connected cross section see reference 134; for an excellent discussion of the formulation of the variational problem see the complementary references 15 and 17.

** For example, as the search for the conjugate torsion function ψ that satisfies $\nabla^2\psi = 0$ over the right cross section S of the twisted prism and satisfies $\psi = [(x^2 + y^2)/2 + \textit{an arbitrary constant}]$ on the boundary s of S.

for the elastic energy per unit length $U \equiv (\tau M/2)$ subject to the conditions: (*a*) the twisting couple $M = 2\mu\tau \int_S \Psi \, dS$ is a prescribed constant; (*b*) Ψ = *an arbitrary constant* (commonly taken as zero) on the entire boundary s. In equation (10–2) the surface integral covers the cross-section S undergoing torque.

An alternative, and somewhat more convenient statement of the problem is to find Ψ that minimizes the integral expression * of (10–3)

$$I(\Psi) = \left(\frac{\mu\tau^2}{2}\right)\int_S (\text{grad}^2 \Psi - 4\Psi) \, dS \tag{10–3}$$

subject to the boundary condition $\Psi = 0$ on s. And yet other formulations of the variational problem are to be found (in particular, see references 15 and 17).

Connection between the Variational and Boundary-Value Problems. The tie between the two methods of interpreting the torsion problem — as a variational problem or as a boundary-value problem in partial differential equations — lies in this: *The Ψ which solves the variational problem necessarily satisfies ** a characteristic partial differential equation (also termed the Euler equation); this equation is precisely the partial differential equation occurring in the boundary-value problem; and inasmuch as the boundary condition is the same in both problems — necessarily so,*** in fact — it follows that this Ψ is also the same one which solves the boundary-value problem.*

Ritz's Method Yields an Approximation. In general, Ritz's method does not yield the function Ψ that exactly minimizes the integral expression $I(\Psi)$. Rather, it yields an n^{th} approximation Ψ_n to Ψ — where Ψ_n is expressed as a linear combination of n functions of a certain type **** individually satisfying the boundary condition, $\Psi = 0$ on s — and an n^{th} approximation $I_n(\Psi_n)$ to I. But *often* the approximation to the true minimum I can be obtained to any desired degree

* Note that by virtue of the quantities defined in the foregoing paragraph $I(\Psi) = U(\Psi) - M\tau$; and since M is a prescribed constant, $I \equiv -\tau M/2 = -U$, wherein I and U designate the minimums of $I(\Psi)$ and $U(\Psi)$.

** A. R. Forsyth, *Calculus of Variations* (University Press, Cambridge, 1927), p. 551.

*** This may not be the case in other problems. See reference 17 and the extension of this discussion in Chapter 3 of reference 85; also H. Bateman, *Partial Differential Equations of Mathematical Physics* (University Press, Cambridge, 1932), p. 157.

**** In general, the approximating function Ψ_n for a singly-connected region is expressed as the sum of a function satisfying the boundary conditions and a linear combination of other functions each of which vanishes on the boundary. In the case of the torsion problem all of the functions satisfy the boundary condition because $\Psi = 0$ on s.

in a comparatively simple manner. The approximation to the true minimum often constantly improves as the terms of the assumed series increase in number. If an appropriate selection of functions has been made — commonly this selection is determined by experience, by the particular coordinate system employed, and by the fact that the integrand stemming from substitution of Ψ_n must be mathematically tractable — the approximation is often quite good when the number of terms in Ψ_n is small. In general, the security for Ψ_n and I_n being actual approximations to Ψ and I is ensured by satisfaction of the conditions that stem from setting the problem as one in the calculus of variations. Specifically, exact conditions on Ψ sufficient to ensure the approach of Ψ_n and I_n to Ψ and I have been the subject of research by Ritz, by Trefftz, and by others.* Provided these sufficient conditions are satisfied, they have shown that

$$\lim_{n\to\infty} \Psi_n = \Psi, \quad \lim_{n\to\infty} I_n(\Psi_n) = I, \quad \text{and} \quad I_n(\Psi_n) \geq I. \tag{10–4}$$

These expressions refer, of course, to equation (10–3).

Trefftz's Procedure. Now though Ritz's method yields a solution where classical methods fail, and though by virtue of the inequality $I_n \geq I$ this solution is an upper limit to the exact solution, *it commonly ensues that the degree of approximation to the exact solution is unknown.* In appreciation of this difficulty Trefftz, his interest aroused by a paper of Courant, has developed a method, akin to Ritz's, but so designed that $I'_n\ (\equiv I_n)$ approaches the true minimum I *from below.* As with Ritz's method, the approximations to Ψ and to I can be carried to any degree of accuracy providing Ψ_n satisfies certain known conditions,

$$\lim_{n\to\infty} \Psi_n = \Psi, \quad \lim_{n\to\infty} I'_n(\Psi_n) = I, \tag{10–5}$$

where $I'_n(\Psi_n) \leq I$. Thus, by application of both Ritz's and Trefftz's methods the true minimum is bounded in known fashion, $I'_n \leq I \leq I_n$. Also the sign and magnitude of the maximum possible deviation from I of I_n, I'_n, or of their arithmetic mean (possibly a closer approximation) can be determined.

Trefftz has developed the theory of his method in two memoirs. One,[7] essentially mathematical, is devoted to elaboration of the gen-

* For a number of references to these purely formal investigations see the bibliography of reference 17; see also N. Kryloff and J. Tamarkin, Bull. Acad. Sci. Russ. [6] **12**, 69–88 (1918); E. Trefftz, *Handbuch der Physik* (J. Springer, Berlin, 1928), Vol. **6**, pp. 130–132.

eral theory of his method. The other,[3] essentially expository, comprises solution of the torsion problem for certain cross sections: * Ψ_1 and M_1 of a [prism of] square [cross section] are obtained by both Trefftz's method and Ritz's, and the two solutions are compared in various fashions. A device for extending both methods to provide solutions for cross sections having re-entrant angles is outlined. By both methods M_1 is effected for an angle-iron with identical flanges.

Comparison of Ritz's and Trefftz's Procedures. Consideration of solutions obtained by both methods indicates that, in general, certain advantages attend the use of Trefftz's rather than Ritz's method. Thus — as Trefftz professes — although the approximating functions used in the latter need not necessarily satisfy the partial differential equation defining the problem, they must satisfy the boundary conditions — not always a simple thing to ensure. Contrariwise, Trefftz's method — to be characterized as a generalization of the classical method of development in terms of particular solutions — requires only a choice of functions that satisfy the partial differential equation, commonly a simpler demand. Secondly, the choice of functions in Trefftz's method is usually such that determination of the maximum error in the solution is easier. Finally, proof of convergence of the approximating function to the true minimum is generally simpler. One considerable advantage, though, attends Ritz's method: ordinarily, the approximation converges somewhat faster than in Trefftz's method. Thus, Holl and Anderson,[9] using both methods to obtain values ** of M_2 for the cross section bounded by arcs of two confocal parabolas find that though both values are in good agreement with the known exact solution, the value afforded by Ritz's method is in closer agreement.

Friedrichs' Procedure. A second method of obtaining a lower bound to the minimum function I solving a given problem (one in some ways more general than that of Trefftz), has been outlined by Friedrichs.[10] It consists of associating with the given minimum problem the auxiliary problem of finding the maximum of a second integral, obtained from the first by a simple transformation and of such nature that its maximum is identical with the minimum of the first integral. As, obviously, the negative of the maximum sought is a minimum, it

* M_1 indicates a first approximation to M, obtained here by using the first approximation Ψ_1 for Ψ.

** Statement of an n^{th} approximation implies determination of the preceding $n - 1$ approximations.

can be obtained in a fashion akin to that of Ritz. The actual procedure can differ in that if only the approximation I_n to the extremum I is sought, and not, as well, the approximation Ψ_n to Ψ producing I, the approximating functions need not *necessarily* satisfy the boundary conditions as they must in the Ritz process. On the other hand, convergence to the minimal solution is somewhat slower.

The explicit theory essential to application of Friedrichs' method has been developed by Friedrichs himself and by Basu.[11,12] The latter treats in detail an important topic barely touched by Friedrichs: namely, determination of the [torsion] function ϕ, the complex conjugate of ψ. If Ψ_n is obtained by Ritz's method, ψ_n follows immediately from the relationship $\psi = \Psi + (x^2 + y^2)/2$. But though ϕ is the complex conjugate of ψ, ϕ_n cannot be obtained from ψ_n by the usual methods of function theory: for the terminated series expressing Ψ_n does not, in general, satisfy the differential equation of the problem $\nabla^2\Psi = -2$, even though the conditions on Ψ_n are such that the function represented by the limit of the series as $n \to \infty$ does satisfy the equation term by term (an interpretation of this latter mathematical phenomenon has been advanced by Goldsbrough [13]). Whence it follows that ψ_n obtained from Ψ_n is not a harmonic function. Friedrichs merely indicates how determination of ϕ conjugate to ψ can be expressed as a variational problem. Basu develops in detail the required analysis to the end that ϕ is obtained through determination by Ritz's method of the function J maximizing a certain double integral expression. In addition, the approximation to this integral expression furnishes the desired lower bound on I, or $J_n \leq I \leq I_n$. Further, by virtue of the identity $I \equiv -\tau M/2$, bounds are provided on M: that is, $I_n \leq \tau M/2 \leq J_n$.

Applications to the Torsion Problem. The details of the analysis contained in Basu's memoirs are exemplified by application to the torsion problem of the square. In the first memoir [11] Ψ_2 is effected by Ritz's method; M_2 by both Ritz's and Friedrichs' methods. In the second memoir [12] ϕ_2 is obtained by Ritz's method; and from ϕ_2 another value of M_2. Close agreement of this last value of M_2 with the known value is taken as evidence that the corresponding expression for ϕ_2 is sufficiently accurate for most purposes; that is, to give the longitudinal displacements of points of the initially plane cross section.

Recently, Diaz and Weinstein [14] have derived the methods of Rayleigh-Ritz and Trefftz by a simple and direct application of Schwarz's inequality and Green's formula. In a companion paper [15] these latter

are employed to obtain upper and lower bounds for M of a single or multiply-connected cross section in terms of a stream function $K(x,y)$ and a harmonic function $f(x,y)$, each of which satisfies certain boundary conditions. The essential advantage of this formulation lies in that the function K satisfies simpler boundary conditions than does the corresponding functions to be employed if the Ritz-Rayleigh method were to be utilized.

In illustration of their analysis, upper and lower bounds on M_1 are determined for a hollow square. These same values, and yet others, were earlier obtained by Weber [16] to illustrate his application of a variational method of elasticity for obtaining upper and lower bounds for the M of a doubly-connected cross section. In addition, Weber obtained M_2 by Ritz's method. Courant,[17] also, has obtained values of M_1 and M_2 for the same cross section by using Ritz's method to solve approximately a variational formulation of the torsion problem somewhat different from the two stated above.

Lack of Knowledge of the Degree of Approximation. At this point we must mention a certain defect in the methods discussed above. Application of them enables estimation of the error in an approximation to the minimum I of the integral expression $I(\Psi)$ because these methods furnish upper and lower bounds on I. Often, however, what is desired, yet usually cannot be obtained (inasmuch as the mathematical technique is lacking) is knowledge of the error in the approximations afforded by Ψ_n to Ψ or to its derivatives. In the torsion problem, for instance, the longitudinal displacement and the stress distribution, respectively furnished by ϕ and by the derivatives of Ψ, are of greater immediate value in practical design than is either Ψ or M afforded by I. But though, as just remarked, a general technique is not available for calculating the error in Ψ_n, it happens that a specialized technique applicable to the torsion problem is known.

Means of Determining the Degree of Approximation. In a paper devoted to exposition of his "thickness parameter" method, the English aeronautical engineer, W. J. Duncan [18] outlines a method * of calculating the maximum deviation between the values of a function Ψ, defined by the conditions that it satisfy $\nabla^2\Psi = -2$ over S and vanish on s, and those of an approximation Ψ_n. Thus, if ϵ' and ϵ'' are the maximum and minimum values within S of a function ϵ defined by $\nabla^2\Psi_n + 2 = \epsilon$, then $(1 - \epsilon''/2) \leq \Psi_n/\Psi \leq (1 - \epsilon'/2)$; and conjunc-

* Obviously, this method is directly extensible to the more general equation, $\nabla^2\Psi =$ *an arbitrary constant.*

tion of this knowledge with the maximum values of ϵ *on the boundary* enables determination of the maximum deviation of Ψ_n from Ψ at any point within or on the boundary of S.

Brachkovsky [19] has advanced a simple method, based on the use of Green's function, which enables evaluation of the maximum value of a function which satisfies Poisson's equation over a region and vanishes on the boundary of the region. Thus, his method can be applied to investigate the accuracy of an approximate value for Ψ.

These procedures, however, do not furnish — at least directly — information on the derivatives of Ψ_n; nor of ϕ, the conjugate of $\psi = \Psi + (x^2 + y^2)/2$. Lacking means to obtain such information, it is tempting to conjecture that if I is bounded closely by J_n and I_n, the corresponding expressions for ϕ_n, Ψ_n, and the derivatives of the latter should yield values in good agreement with the exact values. Though such inference is as yet unsubstantiated by rigorous analysis, empirical support might derive from comparison of values so obtained with exact values calculated from known rigorous solutions (see reference 133 for discussion of more than 100 such solutions).

In a recent paper of which several pages are devoted to a qualitative discussion of the fundamental concepts of the Ritz process and of its merits and defects as regards actual use — which discussion complements the foregoing remarks — Courant [17] discusses these shortcomings and suggests a possible partial remedy through use of "sensitized" integral expressions.

Some Solutions by Ritz's Method. Finally, we note several applications of the Ritz analysis to the torsion problem for certain definite shapes. Dassen [20] has obtained Ψ_6 for both the square and the rectangle and has calculated therefrom expressions for the stresses. Numerical values of maximum surface shearing stress are found to be in good agreement with those calculated from the rigorous solution.

Approximating the normal cross section of a propeller blade by the area bounded by arcs of two nonorthogonal parabolas, Leibenzon [21] obtains Ψ_1 and a first approximation T_{m_1} to the maximum shearing stress T_m. For an area bounded by one of the parabolas and by a line segment normal to its axis he obtains Ψ_2 and T_{m_2}; and for a streamline section akin to an airfoil, Ψ_2, M_2, and T_{m_2}. However, T_{mn} is obtained from Ψ_n by differentiation, in which connection see the third preceeding paragraph.

Duncan, Ellis, and Scruton [22] utilize Ritz's method to investigate the torsion problem for the general isosceles triangle, the exact analytic

solution of which is as yet unknown, though certain special cases have been solved. Ψ_1 and M_1 are obtained. It is found of the former that it is exact for the special case of the equilateral triangle and is a good approximation for isosceles triangles with small included angles, except, possibly, near the base of the triangle; and of the latter that it is "almost certainly amply accurate for practical purposes."

Topaljanski [23] has obtained M_1 and T_{m_1} for a trapezoid of special shape; this yields, as particular cases, the M_1 of a square and of a rectangle with ratio of sides 2/1.

Galerkin's Method. In 1915 Galerkin [24, 25] advanced a variational process which has found considerable use in several fields of applied mechanics, especially in aircraft stress analysis. Galerkin's method has much in common with Ritz's process. Thus, if the differential equation characterizing a given problem is the Euler equation of a variational problem, the approximate solutions by Galerkin's and Ritz's method proceed identically. However, in other problems the two solutions may proceed differently. Specifically, the connections between the two methods have been investigated especially by Romberg,[26] Pearlman,[27] and — at length — Biezeno.[28] Elsewhere, discussing the relative merits of Ritz's, of Galerkin's, and of a third method advanced by Koch [29] and himself, Biezeno [30] indicates a process that encompasses the just-named as special cases. More inclusively, Gross [31, 32] has shown that these procedures and others are derivable as special cases of a very general functional method.

In Galerkin's method an n^{th} approximation to the rigorous solution of a one-dimensional problem is taken in the form

$$\sum_{i=1}^{n} c_i f_i(x), \tag{10-6}$$

wherein the c_i are arbitrary constants and the f_i a selected set of linearly independent functions individually satisfying the boundary conditions. Substitution of this expression in the controlling differential equation (arranged with right-hand member zero) yields "a remainder" or "error" ϵ, a linear function of the n arbitrary constants. Then multiplication of ϵ by each of a set of n arbitrarily chosen, linearly independent functions termed multipliers; integration of each of the products thus formed over the region S of the problem; equation to zero of the resulting expressions; and solution of the resulting set of equations yield a preferred set of values for the arbitrary constants.

Physical Interpretation of the Terms. From the standpoint of mechanics ϵ is interpretable as a generalized force, the arbitrary constants as a set of generalized coordinates, the appropriately chosen multipliers as virtual displacements corresponding to increments of each of these generalized coordinates, the integrated products as weighted means of ϵ, and the vanishing of these weighted means as the vanishing of the virtual work due to the corresponding displacements. In light of this mechanistic interpretation it follows that the degree of accuracy obtained can be increased indefinitely by increasing the number of functions employed and that, in general, exact solution of an m-dimensional problem requires an m-fold infinite set of functions $f_i(x_1, \cdots, x_m)$. However, if the functions and the multipliers are chosen appropriately, it can be anticipated that a good approximation stems from use of only a few functions.

Galerkin's Method Applied to the Torsion Problem. Now in terms of the familiar membrane analogy to the torsion problem, Ψ is proportional to the vertical displacement of the membrane and ϵ is proportional to the applied pressure not balanced by the tensions in the membrane. Accordingly, we represent the stress function by

$$\Psi = \sum_{i=1}^{n} c_i f_i(x,y). \tag{10–7}$$

These functions f_i are appropriately chosen and satisfy the boundary condition $\Psi = 0$ on s. We then substitute Ψ in $\nabla^2\Psi + 2 = 0$ and obtain the "error,"

$$\epsilon = \sum_{i=1}^{n} b_i c_i \tag{10–8}$$

wherein the b_i are known constants. Finally, we choose as multipliers the f_i; integrate each of the products $\epsilon f_i(x,y)$ over S; equate to zero each of the resulting expressions — hence, imposing the condition that the work done by the unbalanced load in a virtual displacement of the membrane proportional to $f_i(x,y)$ be zero; and finally solve the resulting set of algebraic equations for the values c_i.

The most inclusive treatment * in English of Galerkin's method is encompassed in the writings of Duncan.[33–38] For illustration, Duncan [33–35] applies this theory to a variety of problems, among them the torsion problem. Thus, for an airfoil bounded by the loop of a bisym-

* See also a very brief discussion in E. Keller, *Mathematics of Modern Engineering* (Wiley, 1942), Vol. 2, pp. 288–292.

metrical cubic oval, Ψ_2 and M_2 are effected and compared with corresponding values obtained earlier by Duncan by his "thickness parameter" method. Analysis of a bisymmetrical parabolic lenticular cross section is carried out by both methods: Ψ and M are calculated, respectively, to the *4th* and *6th* powers of the thickness parameter; the latter is found to be in close agreement with M_2 obtained by Galerkin's method. Ψ_3 is determined for a rectangle; values of M_3 for squares are in good agreement with values obtained from Saint-Venant's exact solution. Finally, Ψ_1 and M_1 are obtained for an arbitrary area possessing at least one axis of symmetry; corroboratively, M_1 yields the known exact expression for the ellipse.

Kantorovich's and Poritsky's Method. A variational method proposed by Kantorovich,[39, 40] and later by Poritsky,[41] is commensurate in strength and range of application with Ritz's and Galerkin's methods but appears to be little known to English-speaking writers. As exemplified by the torsion problem, Ψ is taken to be of the form

$$\Psi = \sum_{i=1}^{n} X_i Y_i, \tag{10-9}$$

wherein the Y_i are in general assumed functions of both x and y and are such that Ψ satisfies the boundary condition $\Psi = 0$ on s, whereas the X_i are unknown functions of x alone. Then substitution of Ψ in $I(\Psi)$ and imposition of the variational condition that $I(\Psi) = Y_i X_i$ vanishes for all X_i, yields a set of n differential equations, each containing but one X_i. Whence the set of X_i is determined by finding a set of solutions that satisfy boundary conditions determined in part by the assumed form of the Y_i and in part by the original boundary condition.

In this fashion Poritsky obtains Ψ_1, and from it M_1 and T_{m_1}, for a certain rectangle. These values are in excellent agreement with corresponding values calculated from Saint-Venant's exact solution. In similar fashion Ψ_1 is obtained for the ellipse and for the rhomboid.

Further Examples. In Kantorovich's first paper Ψ_1 of a rectangle is obtained. In a succeeding paper Kantorovich shows that for certain types of partial differential equations the system of equations to be solved occur in the form that arises in applying Galerkin's method, whence the general variational procedure can be dispensed with. A technique is then advanced for solving Poisson's equation and the biharmonic equation for domains comprised of several conjoined rectangles. Illustratively, M_1 of a symmetrical angle [iron] is obtained.

Tchepova [42] has utilized Kantorovich's method to obtain Ψ_1 of a symmetrical trapezoid, an angle and an angle with two legs comprised of equal parallelograms, also M_1 for this third figure. Arutinyan,[43] using a special system of curvilinear coordinates, has obtained Ψ_1, M_1 and the stress components for certain thin-walled round-cornered structural shapes of technical importance, i.e., an angle with unequal flanges and a rectangular U. Calculated values are found to be in excellent agreement with available experimental (and other known) values.

In solution of the torsion problem, or yet other two-dimensional problems pertinent to a given area, certain modifications of the above procedure may prove advantageous: say, interchange of the roles of X_i and Y_i; or choice of Y_i as a function of y alone; or utilization of some other permissible variant.

Hovgaard's Method. Yet another variational method is described by Hovgaard.[44, 45] By conjunction of Castigliano's principle and the use of Lagrange's multipliers general variational equations are derived for a prism constrained to simultaneous flexure and torsion. Illustratively, Hovgaard obtains expressions for Ψ, M, and the stress components of an ellipse, quasi-ellipse, equilateral triangle, and rectangle.

Synge's Method. In a recent paper Prager and Synge [46, 47] advance a theory of obtaining approximations in elasticity based on the concept of function space. Illustrative of theory, upper and lower bounds obtained for M and T_m of a square are found to be in excellent agreement with the known exact values. Though not a variational method *per se*, Weinstein * has shown its intimate connection with the variational methods of Ritz and Trefftz.

A Least Squares Method. Let Ψ_n be expressed as a linear combination of a particular solution of $\nabla^2\Psi = -2$ along with $2n$ appropriately chosen harmonic functions. Form the integral $\left[\int_s \Psi_n^2 \, ds\right]$ expressing difference between the squares of the values of Ψ_n and $\Psi(=0)$ on the boundary. Minimization of this integral though variation of the $(2n + 1)$ arbitrary constants of the linear combination yields a set of $(2n + 1)$ algebraic equations which can be solved for the constants. Then as Ψ_n satisfies the differential equation and as Ψ_n is adjusted to best satisfy the boundary condition, it is to be expected that $\Psi_n \approx \Psi$ for values of n sufficiently large.

* See reference 46, pp. 267–269 and reference 14, p. 136.

In this fashion Wiegand [48] obtains Ψ_6 and therefrom M_6, T_{m_6} and the components of stress for a semicircle, and finds values calculated therefrom to be in excellent agreement with corresponding values calculated from the known rigorous solution (also obtained as just outlined by letting $n \to \infty$). With the procedure thus verified, similar expressions are obtained for an arbitrary segment of a circle. From these expressions curves of M and T_m are plotted for a range of geometry. Values from these curves are found to be in agreement with corresponding values obtained by experiment.[49]

Integral Equations Joined with Least Squares. Hildenbrand [50] has extended Crout's method of solution of integral equations by polynomial approximations to enable one to obtain the [illegible] solution of certain integral equations wherein the unknown function, as well as the kernel or Green's function, can have infinite singularities of known order. Approximating the kernel by a linear combination of functions involving a certain number of arbitrary constants and substituting in the integral equation yields an approximation to the known value. Minimizing the square of the difference between the known value and the approximate value through variation of the arbitrary constants yields a set of linear equations for the constants, and hence an approximation to the kernel. This procedure, suitably modified for numerical computation, is illustrated by calculation of excellent approximations to the value of Ψ at the center of, and T_m in, an equilateral triangle.

In yet other fashion Panov [51, 52] has applied Tschaplygin's variational method, for the solution of integral equations, to the torsion problem.

Addenda. Though none of the following papers are concerned with the torsion problem, they are of interest for the reasons noted: by Kryloff [53] and by Kantorovich [39] on Ritz's method — the former contains many references to writings, mostly by Russian authors, not mentioned in the bibliography of reference 17; by Keldych,[54] Williams,[55] Bickley [56] and Frazer, Jones and Skan [57] on Galerkin's method — incidentally, also on the methods of least squares and collocation; by Courant [58] (see also part 4 of reference 17) on the method of gradients — a variational method proclaimed of considerable power, though as yet it has not been employed for numerical solutions; by Hencky [59] and by Goldsbrough [13] on aspects of a certain series method linked with Ritz's method; by Bateman [60] on the connections among various methods.

A number of the variational methods mentioned to this point are treated at length in the books by Kantorovich and Kryloff [61] and by Leibenson.[62, 63] The latter work contains illustrative material on the torsion problem drawn from his earlier writings.[21]

FINITE DIFFERENCE METHODS

Survey. The theory of finite differences affords the most powerful method of finding approximate solutions to boundary value problems which are mathematically intractable to attack by exact analytic methods. In one fashion the general scheme of solution involves:

(*a*) Replacement of the partial differential equations and boundary equations by analogous difference equations

(*b*) Effectance of a sequence of approximate numerical solutions of the difference problem

(*c*) Application of a proper test to determine the convergence of this sequence to the exact solution

(*d*) Determination of limits on the error in the final approximation

In yet another fashion (which has been little used) the given problem is formulated as a variational problem requiring minimization of an integral expression subject to pertinent boundary conditions, these are paralleled by their finite difference analogs, and approximate solution is obtained in an appropriate fashion.*

Though solution of boundary value problems by application of finite difference methods is an idea far from new, much of the pertinent theory is but little known among applied physicists and engineers (as is manifest, for example, by the continued and frequent advance, as new, of analysis which has long existed in the literature). In part, at least, this state of affairs is attributable to the lack of reference texts in English. Thus, though (for example) L. M. Milne-Thomson, *The Calculus of Finite Differences* (University Press, Cambridge, 1933) affords an excellent, modern account of the formal mathematical theory of finite differences, an equally authoritative and inclusive text on the solution of boundary value problems by finite difference procedures is yet to be written.

Perforce, then, one who desires a comprehensive knowledge of the theory and techniques of the finite difference solution of boundary value problems must turn to the periodical literature. Hereof, a logical

* An excellent illustration of this second kind of analysis is to be found in reference 67.

course of study of a given finite difference procedure might well encompass preliminary orientation through reading several articles of a more or less expository nature and subsequent study in chronological order of a sequence of papers wherein development of theory is illustrated by application to one of the classical problems of applied mathematics; for example, the vibrating membrane * or the torsion of a prism.

Finite Differences Applied to the Torsion Problem. The torsion problem for a uniform prism can be considered as the search for the stress function Ψ that satisfies $\nabla^2\Psi = -2$ over the area S, and $\Psi = 0$ on the boundary s. Let S be covered by a net with a regular (or irregular) mesh. Each node point of the net within S is termed a body point of S; each node point that falls on s a boundary point. Then an approximate solution of the torsion problem by finite differences can be considered as the search for a set of discrete numerical values $\Psi(x',y')$ that satisfy the analogous difference equation at each body point (x',y') and have the value 0 at each node point on the boundary. With $\Psi(x',y')$ obtained, the values for ψ, M, T_m and other torsion quantities are derived by numerical integration, numerical differentiation or other appropriate numerical process.

Convergence. The goodness of the solution obviously depends on how well the values thus obtained agree with the exact values. Commonly, it is assumed that if a sufficiently fine mesh is utilized, the approximate values will be in excellent agreement with the exact values and will converge to the exact values as the fineness of the mesh approaches zero. *However, determination of whether or not a sequence of sets of approximate values, effected for a certain boundary value problem in a certain fashion, will converge to the exact values as the fineness of the mesh approaches zero is often a difficult matter to establish.* In general, the custom has been to try the procedure on a problem of which the exact solution is known, and if the values obtained are in good agreement with the known values, to assume that the same procedure will work for other problems. However, proofs of convergence have been established for certain procedures applied to certain boundary value problems, including most of those applicable to the torsion problem.

In general the form of the body and boundary difference equations to be satisfied at a given point depend on the original differential equa-

* For an inclusive bibliography on the eigenvalue problems of the vibrating membrane — and of the wave guide of uniform cross section — see reference 1.

tions, the shape of the boundary, the shape (or shapes) of mesh employed, and the maximum order of differences retained in establishing the difference equations. These variables having been settled, the form of the body and boundary difference equations are fixed, and solution can be effected by one or more well-recognized procedures.

Direct Solution of the Set of Body Equations. Within the writer's knowledge the first solution of the torsion problem by finite differences was effected by Runge,[64] who determined Ψ, and therefrom plotted the stress lines, in a cross-shaped area comprised of five squares. His analysis well illustrates the difficulties of direct solution of the finite difference problem. The nodal values of the function must satisfy the body equations at the node; these equations are linear algebraic equations in the desired nodal values and there are as many equations * as unknown nodal values, whence the set of equations, solved algebraically, yield a unique set of nodal values. But if the mesh is of sufficient fineness, the number of equations is large, and therefore this solution requires a prohibitive amount of work. By use of "joined rectangles" and symmetry, Runge was able to reduce the more than 40 unknowns to 8 after which he solved for these directly. This use of "joined rectangles" has much in common with the use of "block iteration." In fact, tables advanced by Liebmann for use with the former process can be utilized for the latter process.

Richardson's Procedure. Richardson's [65] procedure is exemplified at length in his paper [66] on the analysis of stresses in a dam. His method of "normal functions" is, however, not as simple to use or as easy to grasp as some of the subsequently advanced techniques. In consequence, it has been little used by others. Recently, however, in affecting the finite difference solutions of certain shapes of stepped shafts, Newing [67] has emphasized its usefulness and extended the general theory.

Iteration. In 1918, in a paper frequently cited in the literature Liebmann,[68] after commenting on Runge's and Richardson's work, advanced the procedure of iteration as a more feasible means of effecting the numerical solution of difference problems. This procedure (often termed the Liebmann procedure, though he himself attributes the basic technique to Jacobi) is now the most commonly used finite difference procedure; except possibly in England, where Southwell's relaxation procedure is paramount.

* Since each equation involves the value of the function at a group of related points, such equations are obviously simultaneous.

After sketching the basic analysis Liebmann advances a number of special techniques to be used therewith: assignment of an initial set of node values by linear interpolation of the given boundary values, the use of improved values as the net is traversed, improvement by blocks, conformal mapping, and yet others serve to speed and lighten the course of computation. Subsequently, these suggestions were reworked, improved, extended (and not unoften advanced as new) by many writers in various countries, to the end that comparison of their papers reveals such overlapping and duplication that it is a matter of some difficulty to assign specific improvements in technique to an individual writer.

Frocht [69] has written an excellent account of the bases of certain of the techniques useful in iteration. Comprehensive treatments, particularly of the work of Russian and German writers, are to be found in the books by Mikeladze [70] and Collatz.[71] The expository papers of Emmons [72] and Poritsky [73] are of introductory value.

Laplace's and Poisson's Equations. Gerschgorin [74, 75] has written a series of papers, for the most part in Russian, wherein linear difference equations are applied to obtain the solution of problems involving Laplace's and Poisson's equations. Fortunately, one of the major papers [75] is in German. Therein are discussed methods for calculating the error pertinent to the approximate solutions by iteration of a linear partial differential equation of elliptical type such as the Laplace and Poisson equations. The theory developed in this paper is exemplified by approximation of Ψ for a specifically dimensioned rectangle, determination of the error at a selected point of this rectangle, and check of this latter through comparison with corresponding values determined from the known solution. The analysis of these papers is supplemented and complemented by the subsequent work of Mikeladze [76, 77, 78] and Collatz.[79, 80, 81] In an important memoir [81] where detailed explanation is made of the use of differences of higher order (than the first) to speed up the rate of convergence of the iteration, Ψ for a square is obtained, Gerschgorin's formula for limit of error is applied, and the calculated value at the center is found to be in excellent agreement with the known value.

The papers cited in the preceding paragraph comprise a mine of material of great use to the analyst: equations for calculating the error of the approximation, use of differently shaped and of mixed meshes, the use of higher differences, and yet other topics on which little has

Wolf,[82] Pfeiffer,[83] Kovner,[84] and Courant [85]* for points of general interest from the applied point of view.

Solutions of the Torsion Problem. Shortley and Weller [86] have advanced a scheme of iteration of the differences between the original set of values and the first set of improved values. This has certain advantages of importance to the computer. A succeeding paper [87] gives a solution of M and T for two squares with rounded corners (the sides are equal, the radii different). Moskovitz [88] and Frocht [89] have advanced tables by use of which numerical work is considerably reduced; Frocht and Leven [90,91] a method for selecting a reasonable set of initial values for the unknown.

By iteration Orr [92] developed approximations to M and T_m for a serrated annulus and for a hollow square; in addition, approximations to M for a number of standard structural sections are determined and compared with experimental values. Later Schwalbe,[93] after finding M and T_m for a rectangle in good agreement with corresponding values obtained from the exact solution, determined M of a sharp-cornered channel of uniform thickness and of an I-beam with sloping sides. Such values are compared with corresponding values obtained both from experiment and from various analytic solutions. Other work on structural sections comprises that of Taylor [94] on the surface stresses in a T-beam and of Sunatani and Negoro [95] on the determination of ψ in an angle with equal flanges.

Multiply-connected Cross Sections. Nemenyi [96] appears to have been among the first to discuss the solution of a multiply-connected area by means of finite differences; but he did not illustrate his discussion by solution of a definite problem. Orr's study [92] of the hollow square was followed by Negoro's [97,98] determination of the stress distribution and of T_m in a square pierced by a centrally located circular hole. Values of the latter are in general agreement with corresponding values determined by experiment. Additionally, an estimation of accuracy is afforded by comparison of approximate values calculated for T_m in the annulus with values calculated from the known solution; and by comparing values of T_m for the pierced squares with corresponding values obtained by experiment. More recently Courant [17] ob-

* In a recent paper (reference 17) wherein the fundamental concepts of certain finite difference procedures are briefly but interestingly discussed, Courant mentions a number of papers on finite differences. Many of these papers, though, are on purely formal points of the theory and contain no illustrative examples; however, taken as a third unit in the above scheme of study, they can be read with profit by those desiring the required mathematical background.

tained M for a hollow square and for a square pierced by four symmetrically located smaller hollow squares (the computation for this second case is not given in the reference); and Colin and Newmark,[99] illustrating a method for minimizing the numerical labor incident to the solution of multiply-connected areas, have obtained Ψ on the boundary of the off-centered square hole piercing a larger square.

A Graphical Scheme. A graphical scheme of effecting iteration is described by Panov.[100] As an illustration this method yields values of Ψ for an equilateral triangle, rectangle, and other regular sections which agree (within the range of accuracy of the graphical process) with corresponding values calculated from the known solutions. Again, M of an airfoil is found to be in similar agreement with an earlier numerical determination by Leibenson.

Relaxation. In the course of the past decade Southwell and a group of fellow workers have applied a finite difference procedure commonly termed relaxation (after a physical analogy used early in the development of the theory) to effect the solution of divers types of physical problems. Originally devised by Southwell for the determination of stresses in redundant frameworks — in which connection it has much in common with the earlier Hardy Cross method [101, 102] of distributing moments — it was subsequently developed to the end that Temple,[103] in an investigation of the general theory of the application of relaxation methods to linear systems, remarks "almost all linear systems of equations, algebraic, integral, or differential, can be brought within the scope of relaxation methods, which seem to constitute one of the most powerful methods of computation in mathematical physics and engineering." In America, Grinter's work (see p. 48 for references) has extended moment distribution into a design procedure and has explained how the convergence of joint rotations may be substituted for moment convergence.

Temple's remarks apply equally to iteration. The two procedures have much in common, differing basically in the quantity selected for improvement. Accordingly, a technique developed for one procedure is usually applicable in the other; whence it follows that one should be equally familiar with the literature of both.

Southwell has remarked concerning the philosophy underlying his development of relaxation in several interesting lectures;[104, 105] the considerable work of his group of coworkers is epitomized by his two books[106, 107] (which afford full reference to the original papers). The

theory through the use of physical analogs in the realm of structural analysis, rather than a direct presentation as a finite difference procedure for solving boundary value problems. Accordingly, a detailed study of Southwell's books might well proceed by first reading Frocht's [69] brief, but very clear, account of the basic process, next Shaw's [108] monograph or Fox's paper,[108a] and then the first two chapters of Southwell's second book.* [107]

Convergence of Relaxation Procedure. Investigations of necessary and sufficient conditions ensuring that the approximations stemming from relaxation methods converge to the exact solutions are to be found in papers by Black and Southwell [109] and by Temple.[103] Southwell's books contain ample reference to the original sources of the technique displayed therein. These techniques are supplemented by those detailed in recent papers by Fox,[110, 111] Motz,[112] Bowie,[113] and Shaw.[108]

Solutions of the Torsion Problem by Relaxation. Christopherson and Southwell [114] first illustrated application of relaxation to the torsion problem by determining Ψ and plotting the stress lines in an equilateral triangle pierced by three symmetrically located circular holes. Preliminary to the solution of this problem and indicative of the best procedures to be followed in solving it, Ψ of the equilateral triangle is approximated. An improved technique for dealing with multiply-connected sections has recently been advanced by Shaw [108] and by Colin and Newmark; [99] the former illustrates this technique by determination of Ψ in a hollow square, the latter by finding Ψ on the inner boundary of the figure mentioned above.

Application of relaxation to structural sections is detailed by Christopherson [115] and Shaw.[116, 117] Of an I-beam comprised of three rectangles the former plotted the stress lines and calculated M; the latter did likewise for a standard rolled I-beam. Values of M were found to be in good agreement with values determined by the soap-film experiments of Lyse and Johnston. In a preliminary check of his analysis, Shaw finds values for a rectangle and quadrant of a circle to be in excellent agreement with the known exact values.

Finally, Shaw has calculated M and T_m for a hollow splined shaft; Allen, Southwell and Vaisey [118] have determined the stress lines in a rectangular shaft grooved with two symmetrical keyways.

* The reader interested in the solution of eigenvalue problems by relaxation will find an inclusive list of references in reference 1. Much of this work appeared while Southwell's book was in press. Particular interest accrues to the papers of Motz.

Recently Tranter[118a] has shown how to join the single Fourier transform with relaxation to solve three dimensional boundary value problems with only little more difficulty or computation than is required for two-dimensional problems.

Variational Iteration. Courant[17] has shown how to join finite difference theory with the variational expression of a boundary value problem to the end that a set of difference equations are obtained which yield excellent numerical results with very little numerical calculation. This procedure is illustrated by determination of M for a hollow square and for a square symmetrically pierced by four smaller squares.

Methods for Performing Finite Difference Computation

In recent years considerable attention has been devoted to developing devices for performing the calculation essential to application of a finite-difference procedure. For the most part these comprise application of: punched-card machines; electric calculating boards; mechanical arrangements of racks, gears, springs, etc.; and also graphical approaches.

Punched-Card Machines. Akushsky[119, 120, 121] has outlined several approaches. Kormes and Kormes[122, 123] illustrate their general technique by solution of $\nabla^2\Psi = 0$ over a region with assigned boundary values. The value of punched-card computation is strikingly demonstrated by Bergmann's[124] inclusive solution of the torsion problem for a symmetrical eight-sided figure (roughly, a square with grooved, rounded corners) wherein punched-card machines are used to carry out the great amount of numerical calculation incident to approximate evaluation of the integrals involved in use of his "method of orthogonal functions."

Electric Networks. Nearly twenty years ago Gerschgorin[125] suggested performing iteration of Laplace's equation by use of a d-c calculating board comprised of a network of resistors. More recently, Gutenmacher[126, 127] suggested solution of second order partial differential equations in general by use of an a-c network of resistors, inductors, and capacitors. Kron[128] and a group of fellow workers have used this scheme to obtain by equivalent circuits the solution of a variety of physical problems. As iteration, relaxation, and other procedures can be performed on an a-c calculating board, and as a problem can be solved electrically with considerable precision in but a fraction of the time required for direct numerical computation, an a-c calculating board is particularly useful for rapidly producing ap-

proximate solutions for use in the preliminary analysis of a complex analytical problem. An electric calculating board is commonly considered an analog computer. Electronic computers that compute in the sense of a calculating machine are also available in limited numbers.

Mechanical Methods. Bruk [129] described a rack and gear machine for solving Poisson's or Laplace's equation. Bernstein's [130] description of a generalized Galton apparatus for solving the same equations is illustrated by determination of Ψ and M for a T-section.

Graphical Methods. Runge [131] outlines a method of solving a Dirichlet problem by graphic integration of an integral, the integrand of which is a function of the Green's function for the problem. His student, Rottsieper,[132] has applied this to determine Ψ, M and T of the cross-shaped area comprised of five equal squares. His values proved to be in excellent agreement with Runge's [64] earlier finite difference solution. Independent checks can be afforded by obtaining the Green's function by finite differences or by a method involving Schwarz's alternating process. Panov's [100] method of graphical analysis of difference equations is cited above.

Other Approximate Methods

In addition to variational and finite difference procedures there exist a number of other means of obtaining approximate solutions of physical problems. Those which have been applied to the torsion problem are discussed at length elsewhere.[133, 134, 135] To be added to the papers mentioned in reference 133 are: Lechner [136] and Fuente's [137] discussion of the use of "curvilinear chequers," the latter illustrating his analysis by solving the torsion problem of the equilateral triangle; Cicala's [138, 139] method for thin singly-connected regions (akin to that of Duncan's "thickness parameter" method) illustrated by finding M and T_m for a semicircle and for an area similar to that of a turbine blade; and Parkhomovsky's [140] method of perturbation of the boundary whereby for an area differing not greatly in contour from that for which the solution of the torsion problem is known, the torsion problem can be solved from knowledge of this known solution. This last interesting method is demonstrated by determining expressions for ψ, M and T of areas stemming from the circle, an ellipse (yielding an airfoil), and the annulus. Reference 135 contains account of the solution of the torsion problem for nonuniform, axially symmetric prisms by finite difference procedures.

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